S.-T. Yau College Student Mathematics Contests 2010

Analysis and Differential Equations

Individual

(Please select 5 problems to solve)

1.

a) Let x_k, $k = 1, \dots, n$ be real numbers from the interval $(0, \pi)$ and define $x = \dfrac{\sum_{i=1}^{n} x_i}{n}$. Show that

$$\prod_{k=1}^{n} \frac{\sin x_k}{x_k} \leq \left(\frac{\sin x}{x} \right)^n .$$

b) From

$$\int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2},$$

calculate the integral $\int_0^\infty \sin(x^2) dx$.

2. Let $f : \mathbb{R} \to \mathbb{R}$ be any function. Prove that the set of points x in $\mathbb{R}$ where f is continuous is a countable intersection of open sets.

3. Consider the equation $\dot{x} = -x + f(t,x)$, where $|f(t,x)| \leq \phi(t)|x|$ for all $(t,x) \in \mathbb{R} \times \mathbb{R}$, $\int^\infty \phi(t) dt < \infty$. Prove that every solution approaches zero as $t \to \infty$.

4. Find a harmonic function f on the right half-plane such that when approaching any point in the positive half of the y-axis, the function has limit 1, while when approaching any point in the negative half of the y-axis, the function has limit -1.

5. Let $K(x,y) \in C([0,1] \times [0,1])$. For all $f \in C[0,1]$, the space of continuous functions on $[0,1]$, define a function

$$Tf(x) = \int_0^1 K(x,y) f(y) dy$$

Prove that $Tf \in C([0,1])$. Moreover $\Omega = \{Tf \mid ||f||_{sup} \leq 1\}$ is precompact in $C([0,1])$, i.e. every sequence in Ω has a converging subsequence, here $||f||_{sup} = \sup\{|f(x)| | x \in [0,1]\}$.

6. Prove the Poisson summation formula:

$$\sum_{n=-\infty}^{\infty} f(x + 2\pi n) = \frac{1}{2\pi} \sum_{-\infty}^{\infty} \hat{f}(k) e^{ikx}$$

for all f of functions over $\mathbf{R}$ in the Schwartz space:

$$\mathcal{S} = \{f : (1+x^2)^m |f^{(n)}(x)| \leq C_{m,n}, m, n \geq 0\}$$

where $\hat{f}(\xi) = \int_{\mathbf{R}} f(x) e^{-ix\xi} dx$.

S.-T. Yau College Student Mathematics Contests 2010

Applied Math., Computational Math., Probability and Statistics

Individual

(Please select 5 problems to solve)

1. Let $Z_1, \cdots, Z_n$ be i.i.d. random variables with $Z_i \sim N(\mu, \sigma^2)$. Find

$$E(\sum_{i=1}^{n} Z_i | Z_1 - Z_2 + Z_3).$$

2. Let $X_1, \cdots, X_n$ be pairwise independent. Further, assume that $EX_i = 1 + i^{-1}$ and that $\max_{1 \le i \le n} E|X_i|^{1+\epsilon} < \infty$ for some $\epsilon > 0$. Show that

$$\frac{1}{n} \sum_{i=1}^{n} X_i \xrightarrow{P} 1.$$

3. Let $Z_1, \cdots, Z_6$ be i.i.d. random variables with $Z_i \sim N(0,1)$. Set

$$U^2 = \frac{(Z_1Z_2 + Z_3Z_4 + Z_5Z_6)^2}{Z_2^2 + Z_4^2 + Z_6^2}, \quad V^2 = \frac{U^2(Z_2^2 + Z_4^2)}{U^2 + Z_6^2}.$$

Find and identify the densities of U^2 and V^2.

4. Suppose that three characteristics in a large propulation can be observed according to the following frequencies

$$p_1 = \theta^3, \quad p_2 = 3\theta(1-\theta), \quad p_3 = (1-\theta)^3,$$

where $\theta \in (0,1)$. Let $N_j,\ j = 1,2,3$ be the observed frequencies of characteristic j in a random sample of size n.

(a) Construct the approximate level $(1-\alpha)$ maximum likelihood confidence set for θ.

(b) Derive the asymptotic distribution for the frequency substitution estimator $\hat{\theta}_2 = 1 - (N_3/n)^{1/3}$.

5. (1) Suppose

$$S = \begin{bmatrix} \sigma & \mathbf{u}^T \\ 0 & S_c \end{bmatrix}, \quad T = \begin{bmatrix} \tau & \mathbf{v}^T \\ 0 & T_c \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} \beta \\ \mathbf{b}_c \end{bmatrix},$$

where σ, τ and β are scalars, S_c and T_c are n-by-n matrices, and $\mathbf{b}_c$ is an n-vector. Show that if there exists a vector $\mathbf{x}_c$ such that

$$(S_cT_c - \lambda I)\mathbf{x}_c = \mathbf{b}_c$$

and $\mathbf{w}_c = T_c\mathbf{x}_c$ is available, then

$$\mathbf{x} = \begin{bmatrix} \gamma \\ \mathbf{x}_c \end{bmatrix}, \quad \gamma = \frac{\beta - \sigma\mathbf{v}^T\mathbf{x}_c - \mathbf{u}^T\mathbf{w}_c}{\sigma\tau - \lambda}$$

solves $(ST - \lambda I)\mathbf{x} = \mathbf{b}$.

(2) Hence or otherwise, derive an $O(n^2)$ algorithm for solving the linear system $(U_1U_2 - \lambda I)\mathbf{x} = \mathbf{b}$ where U_1 and U_2 are n-by-n upper triangular matrices, and $(U_1U_2 - \lambda I)$ is nonsingular. Please write down your algorithm and prove that it is indeed of $O(n^2)$ complexity.

(3) Hence or otherwise, derive an $O(pn^2)$ algorithm for solving the linear system $(U_1U_2\cdots U_p - \lambda I)\mathbf{x} = \mathbf{b}$ where $\{U_i\}_{i=1}^p$ are all n-by-n upper triangular matrices, and $(U_1U_2\cdots U_p - \lambda I)$ is nonsingular. Please write down your algorithm and prove that it is indeed of $O(pn^2)$ complexity.

6. (1) Let $A \in \mathbb{R}^{m\times n}$, i.e. A is an m-by-n real matrix. Show that there exists an m-by-m orthogonal matrix U and an n-by-n orthogonal matrix V such that

$$U^TAV = \mathrm{diag}(\sigma_1, \sigma_2, \ldots, \sigma_p),$$

where $p = \min\{m, n\}$ and

$$\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_p \geq 0.$$

(2) Let rank(A)$= r$. Show that for any positive integer $k < r$,

$$\min_{\mathrm{rank}(B)=k} \|A - B\|_2 = \sigma_{k+1}.$$

(*Hint:* Consider the matrix $A_k = \displaystyle\sum_{i=1}^{k} \sigma_i\mathbf{u}_i\mathbf{v}_i^T$, where $\mathbf{u}_i$ and $\mathbf{v}_i$ are columns of U and V respectively.)

S.-T. Yau College Student Mathematics Contests 2010

Geometry and Topology

Individual

(Please select 5 problems to solve)

1. Let $D^* = \{(x, y) \in \mathbb{R}^2 \mid 0 < x^2 + y^2 < 1\}$ be the punctured unit disc in the Euclidean plane. Let g be the complete Riemannian metric on D^* with contsant curvature -1. Find the disctance under the metric between the points $(e^{-2\pi}, 0)$ and $(-e^{-\pi}, 0)$.

2. Show that every closed hypersurface in $\mathbb{R}^n$ has a point at which the second fundamental form is positive definite.

3. Prove that the real projective space $\mathbb{R}P^n$ is orientable if and only if n is odd.

4. Suppose $\pi : M_1 \longrightarrow M_2$ is a C^∞ map of one connected differentiable manifold to another. And suppose for each $p \in M_1$, the differential $\pi_* : T_pM_1 \longrightarrow T_{\pi(p)}M_2$ is a vector space isomorphism.
(a). Show that if M_1 is compact, then π is a covering space projection.
(b). Given an example where M_2 is compact but $\pi : M_1 \longrightarrow M_2$ is not a covering space (but has the π_* isomorphism property).

5. Let Σ_g be the closed orientable surface of genus g. Show that if $g > 1$, then Σ_g is a covering space of Σ_2.

6. Let M be a smooth 4-dimensional manifold. A symplectic form is a closed 2-form ω on M such that $\omega \wedge \omega$ is a nowhere vanishing 4-form.
(a). Construct a symplectic form on $\mathbb{R}^4$.
(b). Show that there are no symplectic forms on S^4.

S.-T. Yau College Student Mathematics Contests 2010

Algebra, Number Theory and Combinatorics

Individual

(Please select 5 problems to solve)

1. Let V be a finite dimensional complex vector space. Let A, B be two linear endomorphisms of V satisfying $AB - BA = B$. Prove that there is a common eigenvector for A and B.

2. Let $M_2(\mathbb{R})$ be the ring of 2×2 matrices with real entries. Its group of multiplicative units is $GL_2(\mathbb{R})$, consisting of invertible matrices in $M_2(\mathbb{R})$.

(a) Find an injective homomorphism from the field $\mathbb{C}$ of complex numbers into the ring $M_2(\mathbb{R})$.

(b) Show that if ϕ_1 and ϕ_2 are two such homomorphisms, then there exists a $g \in GL_2(\mathbb{R})$ such that $\phi_2(x) = g\phi_1(x)g^{-1}$ for all $x \in \mathbb{C}$.

(c) Let h be an element in $GL_2(\mathbb{R})$ whose characteristic polynomial $f(x)$ is irreducible over $\mathbb{R}$. Let $F \subset M_2(\mathbb{R})$ be the subring generated by h and $a \cdot I$ for all $a \in \mathbb{R}$, where I is the idenity matrix. Show that F is isomorphic to $\mathbb{C}$.

(d) Let h' be any element in $GL_2(\mathbb{R})$ with the same characteristic polynomial $f(x)$ as h in (c). Show that h and h' are conjugate in $GL_2(\mathbb{R})$.

(e) If $f(x)$ in (c) and (d) is reducible over $\mathbb{R}$, will the same conclusion on h and h' hold? Give reasons.

3. Let G be a non-abelian finite group. Let $c(G)$ be the number of conjugacy classes in G Define $\overline{c}(G) := c(G)/|G|$, ($|G| = \mathrm{Card}(G)$).

(a) Prove that $\overline{c}(G) \leq \frac{5}{8}$.

(b) Is there a finite group H with $\overline{c}(H) = \frac{5}{8}$?

(c) (open ended question) Suppose that there exists a prime number p and an element $x \in G$ such that the cardinality of the conjugacy class of x is divisible by p. Find a good/sharp upper bound for $\overline{c}(G)$.

4. Let F be a splitting field over $\mathbb{Q}$ the polynomial $x^8 - 5 \in \mathbb{Q}[x]$. Recall that F is the subfield of $\mathbb{C}$ generated by all roots of this polynomial.

(a) Find the degree $[F : \mathbb{Q}]$ of the number field F.
(b) Determine the Galois group $\mathrm{Gal}(F/\mathbb{Q})$.

5. Let $T \subset \mathbb{N}_{>0}$ be a finite set of positive integers. For each integer $n > 0$, define a_n to be the number of all finite sequences $(t_1, \ldots, t_m)$ with $m \leq n$, $t_i \in T$ for all $i = 1, \ldots, m$ and $t_1 + \ldots + t_m = n$. Prove that the infinite series

$$1 + \sum_{n \geq 1} a_n \, z^n \in \mathbb{C}[[z]]$$

is a *rational* function in z, and find this rational function.

6. Describe all the irreducible complex representations of the group S_4 (the symmetric group on four letters).

S.-T. Yau College Student Mathematics Contests 2010

Analysis and Differential Equations

Team

(Please select 5 problems to solve)

1.

a) Let $f(z)$ be holomorphic in D: $|z| < 1$ and $|f(z)| \leq 1$ ($z \in D$). Prove that

$$\frac{|f(0)| - |z|}{1 + |f(0)||z|} \leq |f(z)| \leq \frac{|f(0)| + |z|}{1 - |f(0)||z|}. \qquad (z \in D)$$

b) For any finite complex value a, prove that

$$\frac{1}{2\pi} \int_0^{2\pi} \log |a - e^{i\theta}| d\theta = \max\{\log |a|, 0\}.$$

2. Let $f \in C^1(\mathbf{R})$, $f(x+1) = f(x)$, for all x, then we have

$$||f||_\infty \leq \int_0^1 |f(t)|dt + \int_0^1 |f^{'}(t)|dt.$$

3. Consider the equation

$$\ddot{x} + (1 + f(t))x = 0.$$

We assume that $\int^\infty |f(t)|dt < \infty$. Study the Lyapunov stability of the solution $(x, \dot{x}) = (0, 0)$.

4. Suppose $f : [a, b] \to \mathbf{R}$ be a L^1-integrable function. Extend f to be 0 outside the interval $[a, b]$. Let

$$\phi(x) = \frac{1}{2h} \int_{x-h}^{x+h} f$$

Show that

$$\int_a^b |\phi| \leq \int_a^b |f|.$$

5. Suppose $f \in L^1[0, 2\pi]$, $\hat{f}(n) = \frac{1}{2\pi} \int_0^{2\pi} f(x) e^{-inx} dx$, prove that

1) $\sum_{|n|=0}^{\infty} |\hat{f}(n)|^2 < \infty$ implies $f \in L^2[0, 2\pi]$,

2) $\sum_n |n\hat{f}(n)| < \infty$ implies that $f = f_0, a.e., f_0 \in C^1[0, 2\pi]$,

where $C^1[0, 2\pi]$ is the space of functions f over $[0, 1]$ such that both f and its derivative $f^{'}$ are continuous functions.

6. Suppose $\Omega \subset \mathbf{R}^3$ to be a simply connected domain and $\Omega_1 \subset \Omega$ with boundary Γ. Let u be a harmonic function in Ω and $M_0 = (x_0, y_0, z_0) \in \Omega_1$. Calculate the integral:

$$II = -\int\int_{\Gamma} (u\frac{\partial}{\partial n}(\frac{1}{r}) - \frac{1}{r}\frac{\partial u}{\partial n}) dS,$$

where $\dfrac{1}{r} = \dfrac{1}{\sqrt{(x-x_0)^2+(y-x_0)^2+(z-x_0)^2}}$ and $\dfrac{\partial}{\partial n}$ denotes the out normal derivative with respect to boundary Γ of the domain Ω_1. (Hint: use the formula $\frac{\partial v}{\partial n}dS = \frac{\partial v}{\partial x}dy \wedge dz + \frac{\partial v}{\partial y}dz \wedge dx + \frac{\partial v}{\partial z}dx \wedge dy$.)

S.-T. Yau College Student Mathematics Contests 2010

Applied Math., Computational Math., Probability and Statistics

Team

(Please select 5 problems to solve)

1. Let $X_1, \cdots, X_n$ be independent and identically distributed random variables with continuous distribution functions $F(x_1), \cdots, F(x_n)$, respectively. Let $Y_1 < \cdots < Y_n$ be the order statistics of $X_1, \cdots, X_n$. Prove that $Z_j = F(Y_j)$ has the beta $(j, n-j+1)$ distribution $(j = 1, \cdots, n)$.

2. Let $X_1, \cdots, X_n$ be i.i.d. random variable with a continuous density f at point 0. Let

$$Y_{n,i} = \frac{3}{4b_n}(1 - X_i^2/b_n^2)I(|X_i| \le b_n).$$

Show that

$$\frac{\sum_{i=1}^n (Y_{n,i} - EY_{n,i})}{(b_n \sum_{i=1}^n Y_{n,i})^1/2} \xrightarrow{L} N(0, 3/5),$$

provided $b_n \to 0$ and $nb_n \to \infty$.

3. Let $X_1, \cdots, X_n$ be independently and indentically distributed random variables with $X_i \sim N(\theta, 1)$. Suppose that it is known that $|\theta| \le \tau$, where τ is given. Show

$$\min_{a_1, \cdots, a_{n+1}} \sup_{|\theta| \le \tau} E(\sum_{i=1}^n a_i X_i + a_{n+1} - \theta)^2 = \frac{\tau^2 n^{-1}}{\tau^2 + n^{-1}}.$$

Hint: Carefully use the sufficiency principle.

4. The rules for "1 and 1" foul shooting in basketball are as follows. The shooter gets to try to make a basket from the foul line. If he succeeds, he gets another try. More precisely, he make 0 baskets by missing the first time, 1 basket by making the first shot and xsmissing the second one, or 2 baskets by making both shots.

Let n be a fixed integer, and suppose a player gets n tries at " 1 and 1" shooting. Let N_0, N_1, and N_2 be the random variables recording the number of times he makes 0, 1, or 2 baskets, respectively. Note that $N_0 + N_1 + N_2 = n$. Suppose that shots are independent Bernoulli trails with probability p for making a basket.

(a) Write down the likelihood for (N_0, N_1, N_2).

(b) Show that the maximum likelihood estimator of p is
$$\hat{p} = \frac{N_1 + 2N_2}{N_0 + 2N_1 + 2N_2}.$$
(c) Is $\hat{p}$ an unbiased estimator for p? Prove or disprove. (Hint: $E\hat{p}$ is a polynomial in p, whose order is higher than 1 for $p \in (0,1)$.)
(d) Find the asymptotic distribution of $\hat{p}$ as n tends to ∞.

5. When considering finite difference schemes approximating partial differential equations (PDEs), for example, the scheme

$$u_j^{n+1} = u_j^n - \lambda(u_j^n - u_{j-1}^n) \tag{1}$$

where $\lambda = \dfrac{\Delta t}{\Delta x}$, approximating the PDE

$$u_t + u_x = 0, \tag{2}$$

we are often interested in stability, namely

$$||u^n|| \le C||u^0||, \qquad n\Delta t \le T \tag{3}$$

for a constant $C = C(T)$ independent of the time step Δt and the spatial mesh size Δx. Here $||\cdot||$ is a given norm, for example the L^2 norm or the L^∞ norm, of the numerical solution vector $u^n = (u_1^n, u_2^n, \cdots, u_N^n)$. The mesh points are $x_j = j\Delta x$, $t^n = n\Delta t$, and the numerical solution u_j^n approximates the exact solution $u(x_j, t^n)$ of the PDE (2) with a periodic boundary condition.

(i) Prove that the scheme (1) is stable in the sense of (3) for both the L^2 norm and the L^∞ norm under the time step restriction $\lambda \le 1$.
(ii) Since the numerical solution u^n is in a finite dimensional space, Student A argues that the stability (3), once proved for a specific norm $||\cdot||_a$, would also automatically hold for any other norm $||\cdot||_b$. His argument is based on the equivalency of all norms in a finite dimensional space, namely for any two norms $||\cdot||_a$ and $||\cdot||_b$ on a finite dimensional space W, there exists a constant $\delta > 0$ such that
$$\delta||u||_b \le ||u||_a \le \frac{1}{\delta}||u||_b.$$
Do you agree with his argument? If yes, please give a detailed proof of the following theorem: If a scheme is stable, namely (3) holds for one particular norm (e.g. the L^2 norm), then it is also stable for any other norm. If not, please explain the mistake made by Student A.

6. We have the following 3 PDEs

$$u_t + Au_x = 0, \tag{4}$$

$$u_t + Bu_x = 0, \tag{5}$$

$$(6) \qquad u_t + Cu_x = 0, \qquad C = A + B.$$

Here u is a vector of size m and A and B are $m \times m$ real matrices. We assume $m \geq 2$ and both A and B are diagonalizable with only real eigenvalues. We also assume periodic initial condition for these PDEs.

(i) Prove that (4) and (5) are both well-posed in the L^2-norm. Recall that a PDE is well-posed if its solution satisfies

$$||u(\cdot, t)|| \leq C(T)||u(\cdot, 0)||, \qquad 0 \leq t \leq T$$

for a constant $C(T)$ which depends only on T.

(ii) Is (6) guaranteed to be well-posed as well? If yes, give a proof; if not, give a counter example.

(iii) Suppose we have a finite difference scheme

$$u^{n+1} = A_h u^n$$

for approximating (4) and another scheme

$$u^{n+1} = B_h u^n$$

for approximating (5). Suppose both schemes are stable in the L^2-norm, namely (3) holds for both schemes. If we now form the splitting scheme

$$u^{n+1} = B_h \, A_h u^n$$

which is a consistent scheme for solving (6), is this scheme guaranteed to be L^2 stable as well? If yes, give a proof; if not, give a counter example.

S.-T. Yau College Student Mathematics Contests 2010

Geometry and Topology

Team

(Please select 5 problems to solve)

1. Let $S^n \subset \mathbb{R}^{n+1}$ be the unit sphere, and $\mathbb{R}^n \subset \mathbb{R}^{n+1}$ the equator n-plane through the center of S^n. Let N be the north pole of S^n. Define a mapping $\pi : S^n \backslash \{N\} \to \mathbb{R}^n$ called the stereographic projection that takes $A \in S^n \backslash \{N\}$ into the intersection $A' \in \mathbb{R}^n$ of the equator n-plane $\mathbb{R}^n$ with the line which passes through A and N. Prove that the stereographic projection is a conformal change, and derive the standard metric of S^n by the stereographic projection.

2. Let M be a (connected) Riemannian manifold of dimension 2. Let f be a smooth non-constant function on M such that f is bounded from above and $\Delta f \geq 0$ everywhere on M. Show that there *does not* exist any point $p \in M$ such that $f(p) = \sup\{f(x) : x \in M\}$.

3. Let M be a compact smooth manifold of dimension d. Prove that there exists some $n \in \mathbb{Z}^+$ such that M can be regularly embedded in the Euclidean space $\mathbb{R}^n$.

4. Show that any C^∞ function f on a compact smooth manifold M (without boundary) must have at least two critical points. When M is the 2-torus, show that f must have more than two critical points.

5. Construct a space X with $H_0(X) = \mathbb{Z}$, $H_1(X) = \mathbb{Z}_2 \times \mathbb{Z}_3$, $H_2(X) = \mathbb{Z}$, and all other homology groups of X vanishing.

6. (a). Define the degree $\deg f$ of a C^∞ map $f : S^2 \longrightarrow S^2$ and prove that $\deg f$ as you present it is well-defined and independent of any choices you need to make in your definition.
(b). Prove in detail that for each integer k (possibly negative), there is a C^∞ map $f : S^2 \longrightarrow S^2$ of degree k.

S.-T. Yau College Student Mathematics Contests 2010

Algebra, Number Theory and Combinatorics

Team

(Please select 5 problems to solve)

1. For a real number r, let $[r]$ denote the maximal integer less or equal than r. Let a and b be two positive irrational numbers such that $\frac{1}{a} + \frac{1}{b} = 1$. Show that the two sequences of integers $[ax]$, $[bx]$ for $x = 1, 2, 3, \cdots$ contain all natural numbers without repetition.

2. Let $n \geq 2$ be an integer and consider the Fermat equation

$$X^n + Y^n = Z^n, \qquad X, Y, Z \in \mathbb{C}[t].$$

Find all nontrivial solution (X, Y, Z) of the above equation in the sense that X, Y, Z have no common zero and are not all constant.

3. Let $p \geq 7$ be an odd prime number.

(a) Evaluate the rational number $\cos(\pi/7) \cdot \cos(2\pi/7) \cdot \cos(3\pi/7)$.

(b) Show that $\prod_{n=1}^{(p-1)/2} \cos(n\pi/p)$ is a rational number and determine its value.

4. For a positive integer a, consider the polynomial

$$f_a = x^6 + 3ax^4 + 3x^3 + 3ax^2 + 1.$$

Show that it is irreducible. Let F be the splitting field of f_a. Show that its Galois group is solvable.

5. Prove that a group of order 150 is not simple.

6. Let $V \cong \mathbb{C}^2$ be the standard representation of $SL_2(\mathbb{C})$.

(a) Show that the n-th symmetric power $V_n = \operatorname{Sym}^n V$ is irreducible.

(b) Which V_n appear in the decomposition of the tensor product $V_2 \otimes V_3$ into irreducible representations?

S.-T. Yau College Student Mathematics Contests 2011

Analysis and Differential Equations

Individual

2:30–5:00 pm, July 9, 2011

(Please select 5 problems to solve)

1. a) Compute the integral: $\int_{-\infty}^{\infty} \frac{x \cos x dx}{(x^2+1)(x^2+2)}$,

b) Show that there is a continuous function $f : [0, +\infty) \to (-\infty, +\infty)$ such that $f \not\equiv 0$ and $f(4x) = f(2x) + f(x)$.

2. Solve the following problem:

$$\begin{cases} \frac{d^2u}{dx^2} - u(x) = 4e^{-x}, & x \in (0,1), \\ u(0) = 0, \quad \frac{du}{dx}(0) = 0. \end{cases}$$

3. Find an explicit conformal transformation of an open set $U = \{|z| > 1\} \setminus (-\infty, -1]$ to the unit disc.

4. Assume $f \in C^2[a,b]$ satisfying $|f(x)| \le A, |f''(x)| \le B$ for each $x \in [a,b]$ and there exists $x_0 \in [a,b]$ such that $|f'(x_0)| \le D$, then $|f'(x)| \le 2\sqrt{AB} + D, \forall x \in [a,b]$.

5. Let $C([0,1])$ denote the Banach space of real valued continuous functions on [0, 1] with the sup norm, and suppose that $X \subset C([0,1])$ is a dense linear subspace. Suppose $l : X \to \mathbb{R}$ is a linear map (not assumed to be continuous in any sense) such that $l(f) \ge 0$ if $f \in X$ and $f \ge 0$. Show that there is a unique Borel measure μ on [0, 1] such that $l(f) = \int f d\mu$ for all $f \in X$.

6. For $s \ge 0$, let $H^s(T)$ be the space of L^2 functions f on the circle $T = \mathbb{R}/(2\pi\mathbb{Z})$ whose Fourier coefficients $\hat{f}_n = \int_0^{2\pi} e^{-inx} f(x) dx$ satisfy $\Sigma(1+n^2)^s ||\hat{f}_n|^2 < \infty$, with norm $||f||_s^2 = (2\pi)^{-1}\Sigma(1+n^2)^s|\hat{f}_n|^2$.

a. Show that for $r > s \ge 0$, the inclusion map $i : H^r(T) \to H^s(T)$ is compact.

b. Show that if $s > 1/2$, then $H^s(T)$ includes continuously into $C(T)$, the space of continuous functions on T, and the inclusion map is compact.

S.-T. Yau College Student Mathematics Contests 2011

Applied Math., Computational Math., Probability and Statistics

Individual

6:30–9:00 pm, July 9, 2011

(Please select 5 problems to solve)

1. Given a weight function $\rho(x) > 0$, let the inner-product corresponding to $\rho(x)$ be defined as follows:

$$(f, g) := \int_a^b \rho(x) f(x) g(x) \mathrm{d}x,$$

and let $\|f\| := (f, f)$.

(1) Define a sequence of polynomials as follows:

$$\begin{aligned} p_0(x) &= 1, \quad p_1(x) = x - a_1, \\ p_n(x) &= (x - a_n)p_{n-1}(x) - b_n p_{n-2}(x), \quad n = 2, 3, \cdots \end{aligned}$$

where

$$\begin{aligned} a_n &= \frac{(xp_{n-1}, p_{n-1})}{(p_{n-1}, p_{n-1})}, \quad n = 1, 2, \cdots \\ b_n &= \frac{(xp_{n-1}, p_{n-2})}{(p_{n-2}, p_{n-2})}, \quad n = 2, 3, \cdots . \end{aligned}$$

Show that $\{p_n(x)\}$ is an orthogonal sequence of monic polynomials.

(2) Let $\{q_n(x)\}$ be an orthogonal sequence of monic polynomials corresponding to the ρ inner product. (A polynomial is called *monic* if its leading coefficient is 1.) Show that $\{q_n(x)\}$ is unique and it minimizes $\|q_n\|$ amongst all monic polynomials of degree n.

(3) Hence or otherwise, show that if $\rho(x) = 1/\sqrt{1 - x^2}$ and $[a, b] = [-1, 1]$, then the corresponding orthogonal sequence is the Chebyshev polynomials:

$$T_n(x) = \cos(n \arccos x), \quad n = 0, 1, 2, \cdots .$$

and the following recurrent formula holds:

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x), \quad n = 1, 2, \cdots .$$

(4) Find the best quadratic approximation to $f(x) = x^3$ on $[-1, 1]$ using $\rho(x) = 1/\sqrt{1 - x^2}$.

2. If two polynomials $p(x)$ and $q(x)$, both of fifth degree, satisfy

$$p(i) = q(i) = \frac{1}{i}, \qquad i = 2, 3, 4, 5, 6,$$

and

$$p(1) = 1, \qquad q(1) = 2,$$

find $p(0) - q(0)$.

3. Lay aside m black balls and n red balls in a jug. Supposes $1 \le r \le k \le n$. Each time one draws a ball from the jug at random.

1) If each time one draws a ball without return, what is the probability that in the k-th time of drawing one obtains exactly the r-th red ball?
2) If each time one draws a ball with return, what is the probability that in the first k times of drawings one obtained totally an odd number of red balls?

4. Let X and Y be independent and identically distributed random variables. Show that

$$E[|X + Y|] \ge E[|X|].$$

Hint: Consider separately two cases: $E[X^+] \ge E[X^-]$ and $E[X^+] < E[X^-]$.

5. Suppose that $X_1, \cdots, X_n$ are a random sample from the Bernoulli distribution with probability of success p_1 and $Y_1, \cdots, Y_n$ be an independent random sample from the Bernoulli distribution with probability of success p_2.

(a) Give a minimum sufficient statistic and the UMVU (uniformly minimum variance unbiased) estimator for $\theta = p_1 - p_2$.

(b) Give the Cramer-Rao bound for the variance of the unbiased estimators for $v(p_1) = p_1(1 - p_1)$ or the UMVU estimator for $v(p_1)$.

(c) Compute the asymptotic power of the test with critical region

$$|\sqrt{n}(\hat{p}_1 - \hat{p}_2)/\sqrt{2\hat{p}\hat{q}}| \ge z_{1-\alpha}$$

when $p_1 = p$ and $p_2 = p + n^{-1/2}\Delta$, where $\hat{p} = 0.5\hat{p}_1 + 0.5\hat{p}_2$.

6. Suppose that an experiment is conducted to measure a constant θ. Independent unbiased measurements y of θ can be made with either of two instruments, both of which measure with normal errors: for $i = 1, 2$, instrument i produces independent errors with a $N(0, \sigma_i^2)$ distribution. The two error variances σ_1^2 and σ_2^2 are known. When a measurement y is made, a record is kept of the instrument used so that after n measurements the data is $(a_1, y_1), \ldots, (a_n, y_n)$, where $a_m = i$ if y_m is obtained using instrument i. The choice between instruments is made independently for each observation in such a way that

$$P(a_m = 1) = P(a_m = 2) = 0.5, \quad 1 \le m \le n.$$

Let x denote the entire set of data available to the statistician, in this case $(a_1, y_1), \ldots, (a_n, y_n)$, and let $l_\theta(x)$ denote the corresponding log likelihood function for θ. Let $a = \sum_{m=1}^{n}(2 - a_m)$.

(a) Show that the maximum likelihood estimate of θ is given by

$$\hat{\theta} = \left(\sum_{m=1}^{n} 1/\sigma_{a_m}^2\right)^{-1} \left(\sum_{m=1}^{n} y_m/\sigma_{a_m}^2\right).$$

(b) Express the expected Fisher information I_θ and the observed Fisher information I_x in terms of n, σ_1^2, σ_2^2, and a. What happens to the quantity I_θ/I_x as $n \to \infty$?

(c) Show that a is an ancillary statistic, and that the conditional variance of $\hat{\theta}$ given a equals $1/I_x$. Of the two approximations

$$\hat{\theta} \dot{\sim} N(\theta, 1/I_\theta)$$

and

$$\hat{\theta} \dot{\sim} N(\theta, 1/I_x),$$

which (if either) would you use for the purposes of inference, and why?

S.-T. Yau College Student Mathematics Contests 2011

Geometry and Topology

Individual
9:30–12:00 am, July 10, 2011
(Please select 5 problems to solve)

1. Suppose M is a closed smooth n-manifold.
a) Does there always exist a smooth map $f : M \to S^n$ from M into the n-sphere, such that f is essential (i.e. f is not homotopic to a constant map)? Justify your answer.
b) Same question, replacing S^n by the n-torus T^n.

2. Suppose (X, d) is a compact metric space and $f : X \to X$ is a map so that $d(f(x), f(y)) = d(x, y)$ for all x, y in X. Show that f is an onto map.

3. Let C_1, C_2 be two linked circles in $\mathbb{R}^3$. Show that C_1 cannot be homotopic to a point in $\mathbb{R}^3 \backslash C_2$.

4. Let $M = \mathbb{R}^2/\mathbb{Z}^2$ be the two dimensional torus, L the line $3x = 7y$ in $\mathbb{R}^2$, and $S = \pi(L) \subset M$ where $\pi : \mathbb{R}^2 \to M$ is the projection map. Find a differential form on M which represents the Poincaré dual of S.

5. A regular curve C in $\mathbb{R}^3$ is called a *Bertrand Curve,* if there exists a diffeomorphism $f : C \to D$ from C onto a different regular curve D in $\mathbb{R}^3$ such that $N_xC = N_{f(x)}D$ for any $x \in C$. Here N_xC denotes the principal normal line of the curve C passing through x, and T_xC will denote the tangent line of C at x. Prove that:
a) The distance $|x - f(x)|$ is constant for $x \in C$; and the angle made between the directions of the two tangent lines T_xC and $T_{f(x)}D$ is also constant.
b) If the curvature k and torsion τ of C are nowhere zero, then there must be constants λ and μ such that $\lambda k + \mu\tau = 1$

6. Let M be the closed surface generated by carrying a small circle with radius r around a closed curve C embedded in $\mathbb{R}^3$ such that the center moves along C and the circle is in the normal plane to C at each point. Prove that

$$\int_M H^2 d\sigma \geq 2\pi^2,$$

and the equality holds if and only if C is a circle with radius $\sqrt{2}r$. Here H is the mean curvature of M and $d\sigma$ is the area element of M.

S.-T. Yau College Student Mathematics Contests 2011

Algebra, Number Theory and Combinatorics

Individual

2:30–5:00 pm, July 10, 2011

(Please select 5 problems to solve)

For the following problems, every example and statement must be backed up by proof. Examples and statements without proof will receive no-credit.

1. Let $K = \mathbb{Q}(\sqrt{-3})$, an imaginary quadratic field.

(a) Does there exists a finite Galois extension $L/\mathbb{Q}$ which contains K such that $\mathrm{Gal}(L/\mathbb{Q}) \cong S_3$? (Here S_3 is the symmetric group in 3 letters.)

(b) Does there exists a finite Galois extension $L/\mathbb{Q}$ which contains K such that $\mathrm{Gal}(L/\mathbb{Q}) \cong \mathbb{Z}/4\mathbb{Z}$?

(c) Does there exists a finite Galois extension $L/\mathbb{Q}$ which contains K such that $\mathrm{Gal}(L/\mathbb{Q}) \cong Q$? Here Q is the quaternion group with 8 elements $\{\pm 1, \pm i, \pm j, \pm k\}$, a finite subgroup of the group of units $\mathbb{H}^\times$ of the ring $\mathbb{H}$ of all Hamiltonian quaternions.

2. Let f be a two-dimensional (complex) representation of a finite group G such that 1 is an eigenvalue of $f(\sigma)$ for every $\sigma \in G$. Prove that f is a direct sum of two one-dimensional representations of G

3. Let $F \subset \mathbb{R}$ be the subset of all real numbers that are roots of monic polynomials $f(X) \in \mathbb{Q}[X]$.

(1) Show that F is a field.

(2) Show that the only field automorphisms of F are the identity automorphism $\alpha(x) = x$ for all $x \in F$.

4. Let V be a finite-dimensional vector space over $\mathbb{R}$ and $T : V \to V$ be a linear transformation such that

(1) the minimal polynomial of T is irreducible;

(2) there exists a vector $v \in V$ such that $\{T^i v \mid i \geq 0\}$ spans V.

Show that V contains no non-trivial proper T-invariant subspace.

5. Given a commutative diagram

$$\begin{array}{ccccccccc} A & \to & B & \to & C & \to & D & \to & E \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ A' & \to & B' & \to & C' & \to & D' & \to & E' \end{array}$$

of Abelian groups, such that (i) both rows are exact sequences and (ii) every vertical map, except the middle one, is an isomorphism. Show that the middle map $C \to C'$ is also an isomorphism.

6. Prove that a group of order 150 is not simple.

S.-T. Yau College Student Mathematics Contests 2011

Analysis and Differential Equations

Team

9:00–12:00 am, July 9, 2011

(Please select 5 problems to solve)

1. Let $H^2(\Delta)$ be the space of holomorphic functions in the unit disk $\Delta = \{|z| < 1\}$ such that $\int_\Delta |f|^2 |dz|^2 < \infty$. Prove that $H^2(\Delta)$ is a Hilbert space and that for any $r < 1$, the map $T : H^2(\Delta) \to H^2(\Delta)$ given by $Tf(z) := f(rz)$ is a compact operator.

2. For any continuous function $f(z)$ of period 1, show that the equation

$$\frac{d\varphi}{dt} = 2\pi\varphi + f(t)$$

has a unique solution of period 1.

3. Let $h(x)$ be a C^∞ function on the real line $\mathbb{R}$. Find a C^∞ function $u(x, y)$ on an open subset of $\mathbb{R}$ containing the x-axis such that $u_x + 2u_y = u^2$ and $u(x, 0) = h(x)$.

4. Let $S = \{x \in \mathbb{R} \mid |x - \frac{p}{q}| \le c/q^3, \text{for all } p, q \in \mathbb{Z}, q > 0, c > 0\}$, show that S is uncountable and its measure is zero.

5. Let $sl(n)$ denote the set of all $n \times n$ real matrices with trace equal to zero and let $SL(n)$ be the set of all $n \times n$ real matrices with determinant equal to one. Let $\varphi(z)$ be a real analytic function defined in a neighborhood of $z = 0$ of the complex plane $\mathbb{C}$ satisfying the conditions $\varphi(0) = 1$ and $\varphi'(0) = 1$.

(a) If φ maps any near zero matrix in $sl(n)$ into $SL(n)$ for some $n \ge 3$, show that $\varphi(z) = \exp(z)$.

(b)Is the conclusion of (a) still true in the case $n = 2$? If it is true, prove it. If not, give a counterexample.

6. Use mathematical analysis to show that:

(a) e and π are irrational numbers;

(b) e and π are also transcendental numbers.

S.-T. Yau College Student Mathematics Contests 2011

Applied Math., Computational Math., Probability and Statistics

Team

9:00–12:00 am, July 9, 2011

(Please select 5 problems to solve)

1. Let A be an N-by-N symmetric positive definite matrix. The conjugate gradient method can be described as follows:

$$
\begin{array}{l}
\mathbf{r}_0 = \mathbf{b} - A\mathbf{x}_0, \mathbf{p}_0 = \mathbf{r}_0, \mathbf{x}_0 = 0 \\
\text{FOR } n = 0, 1, \ldots \\
\quad \alpha_n = \|\mathbf{r}_n\|_2^2/(\mathbf{p}_n^T A\mathbf{p}_n) \\
\quad \mathbf{x}_{n+1} = \mathbf{x}_n + \alpha_n \mathbf{p}_n \\
\quad \mathbf{r}_{n+1} = \mathbf{r}_n - \alpha_n A\mathbf{p}_n \\
\quad \beta_n = -\mathbf{r}_{k+1}^T A\mathbf{p}_k/\mathbf{p}_k^T A\mathbf{p}_k \\
\quad \mathbf{p}_{n+1} = \mathbf{r}_{n+1} + \beta_n \mathbf{p}_n \\
\text{END FOR}
\end{array}
$$

Show

(a) α_n minimizes $f(\mathbf{x}_n + \alpha\mathbf{p}_n)$ for all $\alpha \in \mathbb{R}$ where

$$f(\mathbf{x}) \equiv \frac{1}{2}\mathbf{x}^T A\mathbf{x} - \mathbf{b}^T\mathbf{x}.$$

(b) $\mathbf{p}_i^T\mathbf{r}_n = 0$ for $i < n$ and $\mathbf{p}_i^T A\mathbf{p}_j = 0$ if $i \neq j$.

(c) Span$\{\mathbf{p}_0, \mathbf{p}_1, \ldots, \mathbf{p}_{n-1}\}$= Span$\{\mathbf{r}_0, \mathbf{r}_1, \ldots, \mathbf{r}_{n-1}\} \equiv K_n$.

(d) $\mathbf{r}_n$ is orthogonal to K_n.

2. We use the following scheme to solve the PDE $u_t + u_x = 0$:

$$u_j^{n+1} = au_{j-2}^n + bu_{j-1}^n + cu_j^n$$

where a, b, c are constants which may depend on the CFL number $\lambda = \frac{\Delta t}{\Delta x}$. Here $x_j = j\Delta x$, $t^n = n\Delta t$ and u_j^n is the numerical approximation to the exact solution $u(x_j, t^n)$, with periodic boundary conditions.

(i) Find a, b, c so that the scheme is second order accurate.

(ii) Verify that the scheme you derived in Part (i) is exact (i.e. $u_j^n = u(x_j, t^n)$) if $\lambda = 1$ or $\lambda = 2$. Does this imply that the scheme is stable for $\lambda \leq 2$? If not, find λ_0 such that the scheme is stable for $\lambda \leq \lambda_0$. Recall that a scheme is stable if there exist constants M and C, which are independent of the mesh sizes Δx and Δt, such that

$$\|u^n\| \leq Me^{CT}\|u^0\|$$

for all Δx, Δt and n such that $t^n \leq T$. You can use either the L^∞ norm or the L^2 norm to prove stability.

3. Let X and Y be independent random variables, identically distributed according to the Normal distribution with mean 0 and variance 1, $N(0, 1)$.

(a) Find the joint probability density function of $(R,)$, where

$$R = (X^2 + Y^2)^{1/2} \quad \text{and} \quad \theta = \arctan(Y/X).$$

(b) Are R and θ independent? Why, or why not?

(c) Find a function U of R which has the uniform distribution on (0, 1), Unif(0, 1).

(d) Find a function V of θ which is distributed as Unif(0,1).

(e) Show how to transform two independent observations U and V from Unif(0,1) into two independent observations X, Y from $N(0, 1)$.

4. Let X be a random variable such that $E[|X|] < \infty$. Show that

$$E[|X - a|] = \inf_{x \in R} E[|X - x|],$$

if and only if a is a median of X.

5. Let $Y_1, \ldots, Y_n$ be iid observations from the distribution $f(x - \theta)$, where θ is unknown and $f(\ )$ is probability density function symmetric about zero.

Suppose *a priori* that θ has the improper prior $\theta \sim$ Lebesgue (flat) on $(-\infty, \infty)$. Write down the posterior distribution of θ.

Provides some arguments to show that this flat prior is noninformative.

Show that with the posterior distribution in (a), a 95% probability interval is also a 95% confidence interval.

6. Suppose we have two independent random samples $\{Y_1,\ i = 1, \ldots, n\}$ from Poisson with (unknown) mean λ_1 and $\{Y_i,\ i = n+1, \ldots, 2n\}$ from Poisson with (unknown) mean λ_2 Let $\theta = \lambda_1/(\lambda_1 + \lambda_2)$.

(a) Find an unbiased estimator of θ

(b) Does your estimator have the minimum variance among all unbiased estimators? If yes, prove it. If not, find one that has the minimum variance (and prove it).

(c) Does the unbiased minimum variance estimator you found attain the Fisher information bound? If yes, show it. If no, why not?

S.-T. Yau College Student Mathematics Contests 2011

Geometry and Topology

Team
9:00–12:00 am, July 9, 2011
(Please select 5 problems to solve)

1. Suppose K is a finite connected simplicial complex. True or false:
a) If $\pi_1(K)$ is finite, then the universal cover of K is compact.
b) If the universal cover of K is compact then $\pi_1(K)$ is finite.

2. Compute all homology groups of the the m-skeleton of an n-simplex, $0 \le m \le n$.

3. Let M be an n-dimensional compact oriented Riemannian manifold with boundary and X a smooth vector field on M. If $\mathbf{n}$ is the inward unit normal vector of the boundary, show that

$$\int_M div(X)\ dV_M = \int_{\partial M} X \cdot \mathbf{n}\ dV_{\partial M}.$$

4. Let $\mathcal{F}^k(M)$ be the space of all C^∞ k-forms on a differentiable manifold M. Suppose U and V are open subsets of M.
a) Explain carefully how the usual exact sequence

$$0 \longrightarrow \mathcal{F}(U \cup V) \longrightarrow \mathcal{F}(U) \oplus \mathcal{F}V) \longrightarrow \mathcal{F}(U \cap V) \longrightarrow 0$$

arises.
b) Write down the "long exact sequence" in de Rham cohomology associated to the short exact sequence in part (a) and describe explicitly how the map

$$H^k_{deR}(U \cap V) \longrightarrow H^{k+1}_{deR}(U \cup V)$$

arises.

5. Let M be a Riemannian n-manifold. Show that the scalar curvature $R(p)$ at $p \in M$ is given by

$$R(p) = \frac{1}{vol(S^{n-1})} \int_{S^{n-1}} Ric_p(x) dS^{n-1},$$

where $Ric_p(x)$ is the Ricci curvature in direction $x \in S^{n-1} \subset T_pM$, $vol(S^{n-1})$ is the volume of S^{n-1} and dS^{n-1} is the volume element of S^{n-1}.

6. Prove the Schur's Lemma: If on a Riemannian manifold of dimension at least three, the Ricci curvature depends only on the base point but not on the tangent direction, then the Ricci curvature must be constant everywhere, i.e., the manifold is Einstein.

S.-T. Yau College Student Mathematics Contests 2011

Algebra, Number Theory and Combinatorics

Team

9:00–12:00 pm, July 9, 2011

(Please select 5 problems to solve)

For the following problems, every example and statement must be backed up by proof. Examples and statements without proof will receive no-credit.

1. Let F be a field and $\bar{F}$ the algebraic closure of F. Let $f(x, y)$ and $g(x, y)$ be polynomials in $F[x, y]$ such that g.c.d.$(f, g) = 1$ in $F[x, y]$. Show that there are only finitely many $(a, b) \in \bar{F}^{\times 2}$ such that $f(a, b) = g(a, b) = 0$. Can you generalize this to the cases of more than two-variables?

2. Let D be a PID, and D^n the free module of rank n over D. Then any submodule of D^n is a free module of rank $m \leq n$.

3. Identify pairs of integers $n \neq m \in \mathbb{Z}_+$ such that the quotient rings $\mathbb{Z}[x, y]/(x^2 - y^n) \cong \mathbb{Z}[x, y]/(x^2 - y^m)$; and identify pairs of integers $n \neq m \in \mathbb{Z}_+$ such that $\mathbb{Z}[x, y]/(x^2 - y^n) \ncong \mathbb{Z}[x, y]/(x^2 - y^m)$.

4. Is it possible to find an integer $n > 1$ such that the sum

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n}$$

is an integer?

5. Recall that $\mathbb{F}_7$ is the finite field with 7 elements, and $GL_3(\mathbb{F}_7)$ is the group of all invertible 3×3 matrices with entries in $\mathbb{F}_7$.

(a) Find a 7-Sylow subgroup P_7 of $GL_3(\mathbb{F}_7)$.

(b) Determine the normalizer subgroup N of the 7-Sylow subgroup you found in (a).

(c) Find a 2-Sylow subgroup of $GL_3(\mathbb{F}_7)$.

6. For a ring R, let $SL_2(R)$ denote the group of invertible 2×2 matrices. Show that $SL_2(\mathbb{Z})$ is generated by $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. What about $SL_2(\mathbb{R})$?

INDIVIDUAL TEST
S.-T YAU COLLEGE MATH CONTESTS 2012

Analysis and Differential Equations

Please solve 5 out of the following 6 problems,
or highest scores of 5 problems will be counted.

1. Compute the integral

$$\int_0^\infty \frac{x^p}{1+x^2}dx, -1 < p < 1.$$

2. Construct a one to one conformal mapping from the region

$$U = \{z \in \mathbb{C} | |z - \frac{i}{2}| < \frac{1}{2}\} / \{z | |z - \frac{i}{4}| < \frac{1}{4}\}$$

onto the unit disk.

3. Let $f(x)$ be a C^2 function on $\mathbb{R}$. Show that

$$\sup |f'(x)|^2 \leq 4 \sup |f(x)| \sup |f''(x)|.$$

4. Let $f(x)$ be a real measurable function defined on $[a, b]$. Let $n(y)$ be the number of solutions of the equation $f(x) = y$. Prove that $n(y)$ is a measurable function on $\mathbb{R}$.

5. For $1 < p, q < \infty, \frac{1}{p} + \frac{1}{q} = 1$, let g in L^q. Consider the linear functional F on L^p given by: $F(f)$ is equal to the integral of fg. Show that $||F|| = ||g||_q$.

6. Let $\mathbb{R}^n_+ = \{x = (x_1, x_2, ..., x_n) \in \mathbb{R}^n | x_n > 0\}$. Show that the formula

$$u(x) = \frac{2x_n}{n\alpha_n} \int_{\partial\mathbb{R}^n_+} \frac{g(y)}{|x-y|^n} dy, x \in \mathbb{R}^n_+$$

is a solution of the problem

$$\Delta u = 0, \text{ in } \mathbb{R}^n_+, u = g \text{ on } \partial\mathbb{R}^n_+,$$

where α_n is the volume of the unit n dimensional sphere.

INDIVIDUAL TEST
S.-T YAU COLLEGE MATH CONTESTS 2012

Geometry and Topology

Please solve 5 out of the following 6 problems,
or highest scores of 5 problems will be counted.

1. Show that $\pi_3(S^2) \neq 0$.

2. Let M be a smooth manifold of dimension n, and $X_1, \cdots, X_k$ be k everywhere linearly independent smooth vector fields on an open set $U \subset M$ satisfying that $[X_i, X_j] = 0$ for $1 \leq i, j \leq k$. Prove that for any point $p \in U$ there is a coordinate chart (V, y^i) with $p \in V \subseteq U$ and coordinates $\{y^1, \cdots, y^n\}$ such that $X_i = \frac{\partial}{\partial y^i}$ on V for each $1 \leq i \leq k$.

3. Show that any self homeomorphism of $\mathbb{CP}^2$ is orientation preserving.

4. Prove the following version of the isoperimetric inequality: Suppose C is a simple (that is, without self-intersection), smooth, closed curve in the Euclidean plane, with length L. Show that the area enclosed by C is less than or equal to $\frac{L^2}{4\pi}$, and the equality occurs when and only when C is a round circle.

5. Let $x : M \to \mathbb{R}^3$ be a closed surface in 3-dimensional Euclidean space. Its Gaussian curvature and mean curvature are denoted by K and H respectively. Prove that:

$$\iint\limits_M H dA + \iint\limits_M pK dA = 0, \quad \iint\limits_M pH dA + \iint\limits_M dA = 0,$$

where $p = \vec{x} \cdot \vec{n}$ is the support function of M, $\vec{x}$ denotes the position vector of M, $\vec{n}$ denotes the unit normal to M, and dA is the area element of M.

6. Write the structure equation of an orthonormal frame on a Riemannian manifold. Prove the following Riemannian metric g has constant sectional curvature c using the structure equation:

$$g = \frac{\sum_{i=1}^n (dx^i)^2}{[1 + \frac{c}{4} \sum_{i=1}^n (x^i)^2]^2}$$

where $(x^1, \ldots, x^n)$ is a local coordinate system.

INDIVIDUAL TEST

S.-T YAU COLLEGE MATH CONTESTS 2012

Algebra and Number Theory

Please solve 5 out of the following 6 problems,
or highest scores of 5 problems will be counted.

1. Prove that the polynomial $x^6 + 30x^5 - 15x^3 + 6x - 120$ cannot be written as a product of two polynomials of rational coefficients and positive degrees.

2. Let $\mathbb{F}_p$ be the field of p-elements and $GL_n(\mathbb{F}_p)$ the group of invertible n by n matrices.

(1) Compute the order of $GL_n(\mathbb{F}_p)$.
(2) Find a Sylow p-subgroup of $GL_n(\mathbb{F}_p)$.
(3) Compute the number of Sylow p-subgroups.

3. Let V be a finite dimensional vector space over complex field $\mathbb{C}$ with a nondegenerate symmetric bilinear form $(,)$. Let

$$O(V) = \{g \in GL(V) | (gu, gv) = (u, v),\ u, v \in V\}$$

be the orthogonal group. Prove that fixed point subspace $(V \otimes_{\mathbb{C}} V)^{O(n)}$ is 1-dimensional.

4. Let $\mathfrak{D}$ be the ring consisting of all linear differential operators of finite order on $\mathbb{R}$ with polynomial coefficients, of the form

$$D = \sum_{i=0}^{n} a_i(x) \frac{d^i}{dx^i}$$

for some natural number $n \in \mathbb{N}$ and polynomials $a_0(x), \cdots, a_n(x) \in \mathbb{R}[x]$. This ring R operates naturally on $M := \mathbb{R}[x]$, making M a left $\mathfrak{D}$-module.

(1) (to warm up) Suppose that $b(x) \in \mathbb{R}[x]$ is a non-zero polynomial in M, and let $c(x)$ be any element in M. Show that there is an element $D \in \mathfrak{D}$ such that $D(b(x)) = c(x)$.
(2) Suppose that m is a positive integer, $b_1(x), \cdots, b_m(x)$ are m polynomials in M linearly independent over $\mathbb{R}$ and $c_1(x), \cdots, c_m(x)$ are m polynomials in M. Prove that there exists an element $D \in \mathfrak{D}$ such that $D(b_i(x)) = c_i(x)$ for $i = 1, \cdots, m$.

5. Let Λ be a lattice of $\mathbb{C}$, i.e., a subgroup generated by two $\mathbb{R}$-linear independent elements. Let R be the subring of $\mathbb{C}$ consists of elements α such that $\alpha\Lambda \subset \Lambda$. Let $R^\times$ denote the group of invertible elements in R.

(1) Show that either $R = \mathbb{Z}$ or have rank 2 over $\mathbb{Z}$.

(2) Let $n \geq 3$ be a positive integer and $(R/nR)^\times$ the group of invertible elements in the quotient R/nR. Show that the canonical group homomorphism

$$R^\times \to (R/nR)^\times$$

is injective.

(3) Find maximal size of $R^\times$.

6. Let V be a (possible) infinite dimensional vector space over $\mathbb{R}$ with a positive definite quadratic norm $\|\cdot\|$. Let A be an additive subgroup of V with following properties:

(1) $A/2A$ is finite;

(2) for any real number c the set

$$\{a \in A : \qquad \|a\| < c\}$$

is finite.

Prove that A is of finite rank over $\mathbb{Z}$.

INDIVIDUAL TEST
S.-T YAU COLLEGE MATH CONTESTS 2012

Applied Math. and Computational Math.

Please solve 4 out of the following 5 problems.

1. In the numerical integration formula

$$\int_{-1}^{1} f(x)dx \approx af(-1) + bf(c), \tag{1}$$

if the constants a, b, c can be chosen arbitrarily, what is the highest degree k such that the formula is exact for all polynomials of degree up to k? Find the constants a, b, c for which the formula is exact for all polynomials of degree up to this k.

2. Here is the definition of a moving least square approximation of a function $f(x)$ near a point $\overline{x}$ given K points x_k around $\overline{x}$ in $\mathbb{R}$, $k \in [1, \cdots, K]$.

$$\min_{P_{\overline{x}} \in \Pi_m} \sum_{k=1}^{K} |P_{\overline{x}}(x_k) - f_k|^2 \tag{2}$$

where $f_k = f(x_k)$, Π_m is the space of polynomials of degree less or equal to m, i.e.

$$P_{\overline{x}}(x) = \mathbf{b}_{\overline{x}}(x)^T \mathbf{c}(\overline{x}),$$

$\mathbf{c}(\overline{x}) = [c_0, c_1, \cdots, c_m]^T$ is the coefficient vector to be determined by (2),
$\mathbf{b}_{\overline{x}}(x)$ is the polynomial basis vector, $\mathbf{b}_{\overline{x}}(x) = [1, x - \overline{x}, (x - \overline{x})^2, \ldots, (x - \overline{x})^m]^T$.
Assume that there are $K > m$ different points x_k and $f(x)$ is smooth,
(a) prove that there is a unique solution $\overline{P}_{\overline{x}}(x)$ to (2)
(b) denote $h = \max_k \ |x_k - \overline{x}|$, prove

$$|c_i - \frac{1}{i!} f^{(i)}(\overline{x})| = C(f, i) h^{m+1-i}, \ i = 0, 1, \ldots, m,$$

where $f^{(i)}(\cdot)$ is the i-th derivative of f and $C(f, i)$ denote some constant depending on f, i.
(c) if $S = \{x_k | k = 1, 2, \ldots, K\}$ are symmetrically distributed around $\overline{x}$, that is, if $x_k \in S$ then $2\overline{x} - x_k \in S$, prove that

$$|c_i - \frac{1}{i!} f^{(i)}(\overline{x})| = C(f, i) h^{m+2-i}, \ i = 0, 1, \ldots, m,$$

for i ($\in \{0, 1, \cdots, m\}$) with the same parity of m.

3. Describe the forward-in-time and center-in-space finite difference scheme for the one-wave wave equation:

$$u_t + u_x = 0.$$

(i). Conduct the von Neumann stability analysis and comment on their stability property.
(ii). Under what condition on Δt and Δx would this scheme be stable and convergent?
(iii). How many ways you can modify this scheme to make it stable when the CFL condition is satisfied.

4. Let C and D in $\mathbb{C}^{n\times n}$ be Hermitian matrices. Denote their eigenvalues by

$$\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_n \quad \text{and} \quad \mu_1 \ge \mu_2 \ge \cdots \ge \mu_n,$$

respectively. Then it is known that

$$\sum_{i=1}^{n} (\lambda_i - \mu_i)^2 \le \|C - D\|_F^2.$$

1) Let A and B be in $\mathbb{C}^{n\times n}$. Denote their singular values by

$$\sigma_1 \ge \sigma_2 \ge \cdots \ge \sigma_n \quad \text{and} \quad \tau_1 \ge \tau_2 \ge \cdots \ge \tau_n,$$

respectively. Prove that the following inequality holds:

$$\sum_{i=1}^{n} (\sigma_i - \tau_i)^2 \le \|A - B\|_F^2.$$

2) Given $A \in \mathbb{R}^{n\times n}$ and its SVD is $A = U\Sigma V^T$, where $U = (\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n)$, $V = (\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n)$ are orthogonal matrices, and

$$\Sigma = \mathrm{diag}(\sigma_1, \sigma_2, \ldots, \sigma_n), \quad \sigma_1 \ge \sigma_2 \ge \cdots \ge \sigma_n \ge 0.$$

Suppose $\mathrm{rank}(A) > k$ and denote by

$$U_k = (\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k), \quad V_k = (\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k), \quad \Sigma_k = \mathrm{diag}(\sigma_1, \sigma_2, \ldots, \sigma_k),$$

and

$$A_k = U_k \Sigma_k V_k^T = \sum_{i=1}^{k} \sigma_i \mathbf{u}_i \mathbf{v}_i^T.$$

Prove that

$$\min_{\mathrm{rank}(B)=k} \|A - B\|_F^2 = \|A - A_k\|_F^2 = \sum_{i=k+1}^{n} \sigma_i^2.$$

3) Let the vectors $\mathbf{x}_i \in \mathbb{R}^n$, $i = 1, 2, \ldots, n$, be in the space $\mathcal{W}$ with dimension d, where $d \ll n$. Let the orthonormal basis of $\mathcal{W}$ be $W \in \mathbb{R}^{n\times d}$. Then we can represent $\mathbf{x}_i$ by

$$\mathbf{x}_i = \mathbf{c} + W\mathbf{r}_i + \mathbf{e}_i,\ i = 1, 2, \ldots, n,$$

where $\mathbf{c} \in \mathbb{R}^n$ is a constant vector, $\mathbf{r}_i \in \mathbb{R}^d$ is the coordinate of the point $\mathbf{x}_i$ in the space $\mathcal{W}$, and $\mathbf{e}_i$ is the error. Denote $R = (\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_n)$ and $E = (\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n)$. Find W, R and $\mathbf{c}$ such that the error $\|E\|_F$ is minimized.
(*Hint*: write $X = [\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n] = \mathbf{c}(1, 1, \ldots, 1) + WR + E$.)

5. Two primes p and q are called *twin primes* if $q = p + 2$. For example, 5 and 7, 11 and 13, 29 and 31 are twin primes. There is a still unproven (but extensively numerically verified) conjecture that there are infinitely many twin primes and that they are rather common. Show how to factor an integer N which is a product of two twin primes.

INDIVIDUAL TEST
S.-T YAU COLLEGE MATH CONTESTS 2012

Probability and Statistics

Please solve 5 out of the following 6 problems,
or highest scores of 5 problems will be counted.

1. Solve the following two problems:

1) An urn contains b black balls and r red balls. One of the balls was drawn at random, and putted back in the urn with a additional balls of the same color. Now suppose that the second ball drawn at random is red. What is the probability that the first ball drawn was black?

2) Let (X_n) be a sequence of random variables satisfying

$$\lim_{a\to\infty} \sup_{n\geq 1} P(|X_n| > a) = 0.$$

Assume that sequence of random variables (Y_n) converges to 0 in probability. Prove that $(X_n Y_n)$ converges to 0 in probability.

2. Solve the following two problems:

1) Let $(\Omega, \mathcal{F}, P)$ be a probability space, $\mathcal{G}$ be a sub-algebra of $\mathcal{F}$. Assume that X is a non-negative integrable random variable. Set $Y = E[X|\mathcal{G}]$. Prove that

(a)$[X > 0] \subset [Y > 0]$,a.s.;

(b)$[Y > 0] = \text{ess.inf}\{A : A \in \mathcal{G},\ [X > 0] \subset A\}$.

2) Let X and Y have a bivariate normal distribution with zero means, variances σ^2 and τ^2, respectively, and correlation ρ. Find the conditional expectation $E(X|X+Y)$.

3. Suppose that $\{p(i,j) : i = 1, 2, \cdots, m; j = 1, 2, \cdots, n\}$ is a finite bivariate joint probability distribution, that is,

$$p(i,j) > 0,\ \sum_{i=1}^{m}\sum_{j=1}^{n} p(i,j) = 1.$$

(i) Can $\{p(i,j)\}$ be always expressed as

$$p(i,j) = \sum_k \lambda_k a_k(i) b_k(j)$$

for some finite $\lambda_k \geq 0,\ \sum_k \lambda_k = 1,\ a_k(i) \geq 0,\ \sum_{i=1}^m a_k(i) = 1,\ b_k(j) \geq 0,\ \sum_{j=1}^n b_k(j) = 1$?

(ii) Prove or disprove the above relation by use of conditional probability.

4. Let $X_1, \cdots, X_m$ be an independent and identically distributed (i.i.d.) random sample from a cumulative distribution function (CDF) F, and $Y_1, \cdots, Y_n$ an i.i.d. random sample from a CDF G. We want to test $H_0 : F = G$ versus $H_1 : F \neq G$. The total sample size is $N = m + n$. Consider the following two nonparametric tests.

- The Wilcoxon rank sum tests. The test proceeds by first ranking the pooled X and Y samples and then taking the sum of the ranks associated with the Y sample. Let $R_{y_1}, \cdots, R_{y_n}$ be the rankings of the sample $y_1 < \cdots < y_n$ from the pooled sample in increasing order. The Wilcoxon rank sum statistic is defined as $W = \sum_{j=1}^{n} R_{y_j}$.
- The Mann-Whitney U-test. Let $U_{ij} = 1$ if $X_i < Y_j$, and $U_{ij} = 0$ otherwise. The Mann-Whitney U-statistic is defined as $U = \sum_{i=1}^{m} \sum_{j=1}^{n} U_{ij}$. The probability $\gamma = P(X < Y)$ can be estimated as $U/(mn)$. The decision rule is based on assessing if $\gamma = 0.5$.

Assume that there are no tied data values.

(a) Show that $W = U + \frac{1}{2}n(n+1)$, which shows that the two test statistics differ only by a constant and yield exactly the same p-values.

(b) Using the central limit theorem, the Wilcoxon rank sum statistic W can be converted to a Z-variable, which provides an easy-to-use approximation. The transformation is

$$Z_W = \frac{W - \mu_W}{\sigma_W},$$

where μ_W and σ_W^2 are the mean and variance of W under H_0. Show that $\mu_W = \frac{1}{2}n(N+1)$ and $\sigma_W^2 = \frac{1}{12}mn(N+1)$.

5. Let X be a random variable with $EX^2 < \infty$, and $Y = |X|$. Assume that X has a Lebesgue density symmetric about 0. Show that random variables X and Y are uncorrelated, but they are not independent.

6. Let $E_1, \cdots, E_n$ be i.i.d. random variables with $E_i \sim$ Exponential(1). Let $U_1, \cdots, U_n$ be i.i.d. uniformly (on [0,1]) distributed random variables. Further, assume that $E_1, \cdots, E_n$ and $U_1, \cdots, U_n$ are independent.

(a) Find the density of $X = (E_1 + \cdots + E_m)/(E_1 + \cdots + E_n)$, where $m < n$.

(b) Show that $Y = \frac{(n-m)X}{m(1-X)}$ is distributed as the F-distribution with degrees of freedom $(2m, 2(n-m))$

(c) Find the density of $(U_1 \cdots U_n)^{-X}$.

GROUP TEST
S.-T YAU COLLEGE MATH CONTESTS 2012

Analysis and Differential Equations

Please solve 5 out of the following 6 problems.

1. Let $A = [a_{ij}]$ be a real symmetric $n \times n$ matrix. Define $f : \mathbb{R}^n \to \mathbb{R}$ by $f(x_1, \cdots, x_n) = \exp(-\frac{1}{2}\sum_{i,j=1}^n a_{ij}x_ix_j)$. Prove that f is in $L^1(\mathbb{R}^n)$ if and only if the matrix A is positive definite.

Compute $\int_{\mathbb{R}^n} \exp(-\frac{1}{2}\sum_{i,j=1}^n a_{ij}x_ix_j + \Sigma_{i=1}^n b_ix_i)dx$ when A is positive definite.

2. Let V be a simply connected region in the complex plane and $V \neq \mathbb{C}$. Let a, b be two distinct points in V. Let ϕ_1, ϕ_2 be two one-to-one holomorphic maps of V onto itself. If $\phi_1(a) = \phi_2(a)$ and $\phi_1(b) = \phi_2(b)$, show that $\phi_1(z) = \phi_2(z)$ for all $z \in V$.

3. In the unit interval [0, 1] consider a subset
$E = \{x|$ in the decimal expansion of x there is no $4\}$,
show that E is measurable and calculate its measure.

4. Let $1 < p < \infty, L^p([0,1], dm)$ be the completion of $C[0,1]$ with the norm: $||f||p = (\int_0^1 |f(x)|^p dm)^{\frac{1}{p}}$, where dm is the Lebesgue measure. Show that $\lim_{\lambda\to\infty} \lambda^p m(x||f(x)| > \lambda) = 0$.

5. Let $\mathfrak{F} = \{e_\nu\}, \nu = 1, 2, ..., n$ or $\nu = 1, 2, ...$ is an orthonormal basis in an inner product space H. Let E be the closed linear subspace spanned by $\mathfrak{F}$. For any $x \in H$ show that the following are equivalent: 1) $x \in E$; 2) $||x||^2 = \Sigma_\nu |(x, e_\nu)|^2$; 3) $x = \Sigma_\nu (x, e_\nu)e_\nu$.

Let $H = L^2[0, 2\pi]$ with the inner product $< f, g >= \frac{1}{\pi}\int_0^{2\pi} f(x)g(x)dx$,
$\mathfrak{F} = \{\frac{1}{2}, \cos x, \sin x, ..., \cos nx, \sin nx, ...\}$
be an orthonormal basis. Show that the closed linear sub-space E spanned by $\mathfrak{F}$ is H.

6. Let $\mathcal{H} = L^2[0,1]$ relative to the Lebesgue measure and define $(Kf)(s) = \int_0^s f(t)dt$ for each f in $\mathcal{H}$. Show that K is a compact operator without eigenvalues.

GROUP TEST
S.-T YAU COLLEGE MATH CONTESTS 2012

Geometry and Topology

Please solve 5 out of the following 6 problems.

1. Prove that the real projective space $\mathbb{RP}^n$ is a differentiable manifold of dimension n.

2. Let M, N be n-dimensional smooth, compact, connected manifolds, and $f : M \to N$ a smooth map with rank equals to n everywhere. Show that f is a covering map.

3. Given any Riemannian manifold (M^n, g), show that there exists a unique Riemannian connection on M^n.

4. Let S^n be the unit sphere in $\mathbb{R}^{n+1}$ and $f : S^n \to S^n$ a continuous map. Assume that the degree of f is an odd integer. Show that there exists $x_0 \in S^n$ such that $f(-x_0) = -f(x_0)$.

5. State and prove the Stokes theorem for oriented compact manifolds.

6. Let M be a surface in $\mathbb{R}^3$. Let D be a simply-connected domain in M such that the boundary ∂D is compact and consists of a finite number of smooth curves. Prove the Gauss-Bonnet Formula:

$$\int_{\partial D} k_g \, ds \; + \sum_j (\pi - \alpha_j) + \iint_D K \, dA \; = \; 2\pi,$$

where k_g is the geodesic curvature of the boundary curve. Each α_j is the interior angle at a vertex of the boundary, K is the Gaussian curvature of M, and the 2-form dA is the area element of M.

GROUP TEST

S.-T YAU COLLEGE MATH CONTESTS 2012

Algebra and Number Theory

Please solve 5 out of the following 6 problems.

1. Let $a_1, \cdots, a_n$ and $b_1, \cdots, b_n$ be complex numbers such that $a_i + b_j \neq 0$ for all $i, j = 1, \cdots, n$. Define $c_{ij} := \dfrac{1}{a_i + b_j}$ for all $i, j = 1, \cdots, n$, and let C be the $n \times n$ determinant with entries c_{ij}. Prove that

$$det(C) = \frac{\prod_{1 \leq i < j \leq n}(a_i - a_j)(b_i - b_j)}{\prod_{1 \leq i,j \leq n}(a_i + b_j)}.$$

2. Recall that $\mathbb{F}_7$ is the finite field with 7 elements, and $GL_3(\mathbb{F}_7)$ is the group of all invertible 3×3 matrices with entries in $\mathbb{F}_7$.

(1) Find a 7-Sylow subgroup P_7 of $GL_3(\mathbb{F}_7)$.

(2) Determine the normalizer subgroup N of the 7-Sylow subgroup you found in (a).

(3) Find a 2-Sylow subgroup of $GL_3(\mathbb{F}_7)$.

3. Let V be a finite dimensional vector space with a positive definite quadratic form $(-, -)$. Let $O(V)$ denote the orthogonal group:

$$O(V) = \{g \in GL(V) : \qquad (gx, gy) = (x, y), \quad \forall x, y \in V\}.$$

For any non-zero $v \in V$, let s_v denote the reflection on V:

$$s_v(w) = w - 2\frac{(v, w)}{(v, v)}v.$$

(1) Show that $s_v \in O(V)$;

(2) Show that if v and w are vectors in V with $\|v\| = \|w\|$,then there is either a reflection or product of two reflections that takes v into w;

(3) Deduce that every element of the orthogonal group of V can be written as the product of at most $2 \dim V$ reflections.

4. Consider the real Lie group $SL_2(\mathbb{R})$ of 2 by 2 matrices of determinant one. Compute the fundamental group of $SL_2(\mathbb{R})$ and describe the Lie group structure on the universal covering

$$\widetilde{SL}_2(\mathbb{R}) \to SL_2(\mathbb{R}).$$

5. Let $f \in \mathbb{C}[x, y, z]$ be an irreducible homogenous polynomial of degree $d > 0$. For each integer $n \geq d$, define

$$P(n) = \dim_{\mathbb{C}} \mathbb{C}[x, y, z]_n / f \cdot \mathbb{C}[x, y, z]_{n-d}$$

where $\mathbb{C}[x,y,z]_d$ is the subspace of homogenous polynomials of degree n. Show there are constants c such that for n sufficiently large,

$$P(n) = dn + c.$$

6. Let p be an odd prime and $\mathbb{Z}_p$ the p-adic integer which can be defined as the projective limit of $\mathbb{Z}/p^n\mathbb{Z}$ and let $\mathbb{Q}_p$ be its fractional field. Let $\mathbb{Z}_p^\times$ denote the group of invertible elements in $\mathbb{Z}_p$ which is also the projective limit of $(\mathbb{Z}/p^n\mathbb{Z})^\times$.

(1) For any integer a is not divisible by p, show that the sequence $(a^{p^n})_n$ convergent to an element $\omega(a) \in \mathbb{Z}_p$ satisfying

$$\omega(a)^{p-1} = 1, \qquad \omega(a) \equiv a \pmod{p}.$$

Moreover, $\omega(a)$ depends only on $a \mod p$.

(2) Define a logarithmic function log on $1 + p\mathbb{Z}_p$ by usual formula:

$$\log(1+px) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{p^n}{n} x^n.$$

Show that the logarithmic function is convergent and define an isomorphism

$$1 + p\mathbb{Z}_p \to p\mathbb{Z}_p.$$

Moreover, on the dense subgroup $\log(1+p)\mathbb{Z}$, the inverse is given by

$$\log(1+p) \cdot x \mapsto (1+p)^x, \qquad \forall x \in \mathbb{Z}.$$

(3) Deduce from above that $\mathbb{Z}_p^\times \simeq \mathbb{Z}_p \times \mathbb{Z}/(p-1)\mathbb{Z}$.

GROUP TEST
S.-T YAU COLLEGE MATH CONTESTS 2012

Applied Math. and Computational Math.

Please solve 4 out of the following 5 problems.

1. If the function $u(x)$ is in C^{k+1} (has continuous $(k+1)$-th derivative) on the interval $[0,2]$, and a sequence of polynomials $p_n(x)$ $(n = 1,2,3,...)$ of degree at most k satisfies

$$|u(x) - p_n(x)| \le \frac{C}{n^{k+1}} \qquad \forall\ 0 \le x \le \frac{1}{n}, \tag{1}$$

where the constant C is independent of n, prove

$$|u(x) - p_n(x)| \le \frac{\tilde{C}}{n^{k+1}} \qquad \forall\ \frac{1}{n} \le x \le \frac{2}{n},$$

with another constant $\tilde{C}$ which is also independent of n.

2. Consider the one-dimensional elliptic equation

$$-\frac{d^2}{dx^2}u(x) = f(x), \quad 0 < x < 1,$$

with homogeneous boundary condition, $u(0) = 0$ and $u(1) = 0$, $f \in L^2(0,1)$.

(i) Describe the standard piecewise linear finite element method for this boundary value problem.

(ii) Is this method stable and convergent? If so, what is the order of convergence?

(iii). In this case, the linear finite element method has a super convergence property at the nodal point x_j $(j = 1,2,...,N)$, i.e. $u_h(x_j) = u(x_j)$, here u_h is the finite element solution and u is the exact solution. Could you explain why?

3. Let $A = (a_{ij}) \in M_{N\times N}(\mathbb{C})$ be strictly diagonally dominant, that is,

$$|a_{ii}| > \sum_{j=1, j\neq i}^{N} |a_{ij}| \ \text{ for all } 1 \le i \le N,$$

Assume that $A = I + L + U$ where I is the identity matrix, L and U are the lower and upper triangular matrices with zero diagonal entries.

Now, we consider solving the linear system $Ax = b$ by the following iterative scheme:

$$(*) \quad x^{k+1} = (I+\alpha\Omega L)^{-1}[(I-\Omega)-(1-\alpha)\Omega L-\Omega U]x^k+(I+\alpha\Omega L)^{-1}b$$

where $\Omega := \mathbf{diag}(\omega_1, ...\omega_N)$ and $0 \leq \alpha \leq 1$. (When $\alpha = 1$, it gives the SOR method.)

(1) Prove that the linear system $Ax = b$ has a unique solution.
(2) Prove that the necessary condition for the convergence of (*) is

$$\prod_{i=1}^{N} |1-\omega_i| < 1$$

(3) Let $M = (I+\alpha\Omega L)^{-1}[(I-\Omega)-(1-\alpha)\Omega L-\Omega U)]$. Prove that the spectral radius $\rho(M)$ of M is bounded by:

$$\rho(M) \leq \max_i \frac{|1-\omega_i| + |\omega_i|(|1-\alpha|l_i + u_i)}{1-|\omega_i\alpha|l_i}$$

whenever $|\omega_i\alpha|l_i$ for all $1 \leq i \leq N$ where $l_i = \sum_{j<i}|a_{ij}|$ and $u_i = \sum_{j>i}|a_{ij}|$.
(4) Using (c), prove that the sufficient condition for the convergence of (*) is

$$0 < \omega_i < \frac{2}{1+l_i+u_i} \quad \text{for all } 1 \leq i \leq N$$

4. The famous *RSA cryptosystem* is based on the assumed difficulty of factoring integers $N = pq$ (called RSA integers) which are products of two large primes p and q which should be kept secret. Currently p and q are chosen to be about 500 bits long, that is,

$$p, q \approx 2^{500}.$$

Assume someone uses the following algorithm to find secret n-bit primes p and q to form an RSA integer $N = pq$:

- Choose a random odd 500-bit integer s.
- Test the odd numbers s, $s+2$, $s+4$, etc. for primality until the first prime p is found (note the primality testing is very easy nowdays).
- Continue testing $p+2$, $p+4$, $p+6$, etc. for primality until the second prime q is found.
- Compute and publish $N = pq$, but keep p and q secret.

How secure is this procedure? Can you suggest an algorithm to factor an RSA integer $N = pq$ generated this way?

Note that there are about $x/\log x$ primes up to x, where $\log x$ is the natural logarithm. This means that the expected gap between two consecutive n-bit primes is

$$\log 2^n = n \log 2 \approx 0.69 \cdot n.$$

5. The solution $h(r,t)$ of the following Boussinesq equation describes the hight of a circular drop of fluid spreading on a dry surface $h=0$:

$$\frac{\partial h}{\partial t} = \Delta_r(h^2) = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial(h^2)}{\partial r}\right), \quad r>0, \quad t>1$$

with

$$\left.\frac{\partial h}{\partial r}\right|_{r=0} = 0, \quad \int_0^\infty h(r,t)r\,dr \equiv \frac{1}{64}$$

The solution is positive on a finite range $0 \le r \le r_*(t)$ with $h(r_*(t),t) = 0$ defining a moving "edge" position with no fluid outside of the droplet. For $r > r_*(t)$ truncate the solution beyond the edge to be zero ($h \equiv 0$ for $r > r_*(t)$).

- **(a):** Show that this problem is scale invariant by finding relations $h(r,t) = H(T)\tilde{h}(\tilde{r},\tilde{t})$, $r = R(T)\tilde{r}$, $t = T\tilde{t}$ so that the problem for $\tilde{h}(\tilde{r},\tilde{t})$ is identical to the original problem.
- **(b):** Determine the ODE for the similarity function $\Phi(\eta)$ with $h(r,t) = t^\alpha\Phi(\eta)$, $r = \eta t^\beta$.
- **(c):** Determine the explicit solution for $\Phi(\eta)$ and then use $h(r,t) = t^\alpha\Phi(\eta)$ to find $r_*(t)$ for $t \ge 1$.
  Hint $\int_0^\infty hr\,dr = \int_0^{r_*} hr\,dr$.

GROUP TEST
S.-T YAU COLLEGE MATH CONTESTS 2012

Probability and Statistics

Please solve 5 out of the following 6 problems.

1. Let (X_n) be a sequence of i.i.d. random variables.

1) Assume that each X_n satisfies the exponential distribution with parameter 1 (i.e. $P(X_n \geq x) = e^{-x}, x \geq 0$). Prove that

(a) $P(X > \alpha \log n, i.o.) = 0$, if $\alpha > 1$; $P(X > \alpha \log n, i.o.) = 1$, if $\alpha \leq 1$.

Here "i.o" stands for "infinitely often", and $A_n, i.o.$ stands $\limsup_{n\to\infty} A_n$.

(b) Let $L = \limsup_{n\to\infty}(X_n/\log n)$, then $P(L = 1) = 1$.

2) Assume that each X_n satisfies the Poisson distribution with parameter λ (i.e. $P(X_n = k) = \frac{\lambda^k}{k!}e^{-\lambda}, k = 0, 1, 2, \cdots$.) Put

$$L = \limsup_{n\to\infty}(X_n \log\log n/\log n).$$

Prove that $P(L = 1) = 1$.

2. Let X_i be i.i.d exponential r.v with rate one, $i \geq 1$. Let N be a geometric random variable with success probability p, $0 < p < 1$, i.e. $P(N = k) = (1-p)^{k-1}p, k = 1, 2, \cdots$, and independent of all X_i, $i \geq 1$. Find the distribution of $\sum_{i=1}^{N} X_i$.

3. Let X and Y be i.i.d real valued r.v's. Prove that $P(|X+Y| < 1) \leq 3P(|X-Y| < 1)$.

4. Suppose $S = X_1 + X_2 + \cdots + X_n$, a sum of independent random variables with X_i distributed Binomial(1, p_i). Show that $\mathbb{P}(S\ \mathit{even}) = 1/2$ if and only if at least one p_i equals 1/2.

5. Let B_θ denote the closed unit ball in $\mathbb{R}^2$ with center θ. Suppose $X_1, X_2, \cdots, X_n$ are independently distributed on B_θ, for an unknown θ in $\mathbb{R}^2$. Denote that maximum likelihood estimator by $\hat{\theta}$. Show that $|\hat{\theta} - \theta| = O_p(1/n)$.

6. Suppose that $X_1, \cdots, X_n$ are a random sample from the Bernoulli distribution with probability of success p_1 and $Y_1, \cdots, Y_n$ be an independent random sample from the Bernoulli distribution with probability of success p_2.

(a) Derive the maximum likelihood ratio test statistic for

$$H_0 : p_1 = p_2 \longleftrightarrow H_1 : p_1 \neq p_2.$$

(Note: No simplification of the resulting test statistic is required. However, you need to give the asymptotic null.)

(b) Compute the asymptotic power of the test with critical region

$$|\sqrt{n}(\hat{p}_1 - \hat{p}_2)/\sqrt{2\hat{p}\hat{q}}| \geqslant z_{1-\alpha}$$

when $p_1 = p$ and $p_2 = p + n^{-1/2}\Delta$, where $\hat{p} = 0.5.\hat{p}_1 + 0.5\hat{p}_2$.

S.-T. Yau College Student Mathematics Contests 2013

Analysis and Differential Equations

Individual

Please solve 5 out of the following 6 problems.

1. Suppose that f is an integrable function on $\mathbf{R}^d$. For each $\alpha > 0$, let $E_\alpha = \{x||f(x)| > \alpha\}$. Prove that:

$$\int_{\mathbf{R}^d} |f(x)|dx = \int_0^\infty m(E_\alpha)d\alpha.$$

2. Let $p(z)$ be a polynomial of degree $d \geq 2$, with distinct roots $a_1, a_2, \cdots, a_d$. Show that

$$\sum_{i=1}^{d} \frac{1}{p'(a_i)} = 0.$$

3. Let α be a number such that α/π is not a rational number. Show that:

1) $\lim_{N\to\infty} \Sigma_{n=1}^N e^{ik(x+n\alpha)} = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ikt}dt$.

2) For every continuous periodic function $f : \mathbf{R} \to \mathbf{C}$ of period 2π, we have

$$\lim_{N\to\infty} \frac{1}{N}\Sigma_{n=1}^N f(x+n\alpha) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)dt.$$

4. Let u be a positive harmonic function over the punctured complex plane $\mathbf{C}/\{0\}$. Show that u must be a constant function.

5. Suppose $H = L^2(B), B$ is the unit ball in $\mathbf{R}^d$. Let $K(x,y)$ be a measurable function on $B \times B$ that satisfies

$$|K(x,y)| \leq A|x-y|^{-d+\alpha}$$

for some $\alpha > 0$, whenever $x, y \in B$. Define

$$Tf(x) = \int_B K(x,y)f(y)dy$$

(a) Prove that T is a bounded operator on H.

(b) Prove that T is compact.

6. Let A be a $n \times n$ real non-degenerate symmetric matrix. For $\lambda \in \mathbf{R}^+$, we define: $\int_{\mathbf{R}} \exp(i\lambda x^2)dx = \lim_{\epsilon\to 0^+}\int_{-\infty}^{\infty} \exp(i\lambda x^2 - \frac{1}{2}\epsilon x^2)dx$. Show that:

$$\int_{\mathbf{R}^n} \exp(i\frac{\lambda}{2} < Ax, x > -i < x, \xi >)dx$$

$$= (\frac{2\pi}{\lambda})^{n/2} |\det(A)|^{-1/2} \exp(-\frac{i}{2\lambda} < A^{-1}\xi, \xi >) \exp(\frac{i\pi}{4} sgnA).$$

Here $\lambda \in \mathbf{R}^+, \xi \in \mathbf{R}^n, sgn(A) = \nu_+(A) - \nu_-(A), \nu_+(A)(\nu_-(A))$ is the number of positive (negative) eigenvalues of A.

S.-T. Yau College Student Mathematics Contests 2013

Geometry and Topology

Individual

Please solve 5 out of the following 6 problems.

1. Find the homology and fundamental group of the space $X = S^1 \times S^1/\{p, q\}$ obtained from the torus by identifying two distinct points p, q to one point.

2. Suppose (X, d) is a compact metric space and $f : X \to X$ is a map so that $d(f(x), f(y)) = d(x, y)$ for all x, $y \in X$. Show that f is an onto map.

3. Let M^2 ba a complete regular surface and K be the Gaussian curvature. Suppose $\sigma : [0, \infty) \to M$ is a geodesic such that $K(\sigma(t)) \leq f(t)$, where f is a differentiable function on $[0, \infty)$. Prove that any solution $u(t)$ of the equation

$$u''(t) + f(t)u(t) = 0$$

has a zero on $[0, t_0]$, where $\sigma(t_0)$ is the first conjugate point to $\sigma(0)$ along σ.

4. Let g_1, g_2 be Riemannian metrics on a differentiable manifold M, and denote by R_1 and R_2 their respective Riemannian curvature tensor. Suppose that $R_1(X, Y, Y, X) = R_2(X, Y, Y, X)$ holds for any tangent vectors X, $Y \in T_pM$. Show that $R_1(X, Y, Z, W) = R_2(X, Y, Z, W)$ for any X, Y, Z, $W \in T_pM$.

5. Let M^n be an even dimensional, orientable Riemannian manifold with positive sectional curvature. Let $\sigma : [0, l] \to M$ be a closed geodesic, namely, σ is a geodesic with $\sigma(0) = \sigma(l)$ and $\sigma'(0) = \sigma'(l)$. Show that there exist an $\epsilon > 0$ and a smooth map $F : [0, l] \times (-\epsilon, \epsilon) \to M$, such that $F(t, 0) = \sigma(t)$, and for any fixed $s \neq 0$ in $(-\epsilon, \epsilon)$, $\sigma_s(t) = F(t, s)$ is a closed smooth curve with length less than that of σ.

6. Let (M^2, ds^2) be a minimal surface in $\mathbb{R}^3$, where ds^2 is the restriction of the Euclidean metric. Assume that the Gaussian curvature K of (M^2, ds^2) is negative. Denote by $\widetilde{K}$ the Gaussian curvature of the metric $\widetilde{ds^2} = -K ds^2$. Show that $\widetilde{K} = 1$.

S.-T. Yau College Student Mathematics Contests 2013

Algebra and Number Theory

Individual

This exam of 160 points is designed to test how much you know rather than how much you don't know. You are not expected to finish all problems but do as much as you can.

1. (20pt)

1.1 (15 pt) Classify finite groups of order 26 up to isomorphisms.

1.2 (5 pt) For each finite group G of order 26, describe the group $Aut(G)$ of automorphisms of G.

2. (20 pt) Consider $f \in \mathbb{Z}_{>0}$ and nonzero vector spaces V_i indexed by $i \in \mathbb{Z}/f\mathbb{Z}$. Suppose that there are linear maps $\phi_i : V_i \to V_{i+1}$ and $\psi_i : V_i \to V_{i-1}$ such that

$$\phi_{i-1} \circ \psi_i = 0, \quad \psi_{i+1} \circ \phi_i = 0.$$

(We may think of a circular graph with oriented edges such that the "Orpheus condition" holds: *Whenever you turn back while traveling through the graph you are killed.*)

Prove that there exists lines $\ell_i \subset V_i$ for every $i \in \mathbb{Z}/f\mathbb{Z}$ such that

$$\phi_i(\ell_i) \subset \ell_{i+1}, \quad \psi_i(\ell_i) \subset \ell_{i-1}$$

under one of the following two conditions:

2.1 (10 pt) all $\psi_i = 0$, or

2.2 (10 pt) $\dim V_i$ are equal to each other.

3. (20pt) For a parameter $t = (t_0, t_1, \cdots t_5) \in \mathbb{F}_5^6$ with $t_0 \neq 0$ and $\{t_i, i > 0\}$ an ordering of elements in $\mathbb{F}_5$, define a polynomial

$$P_t(x) = (x - t_1)(x - t_2)(x - t_3) + t_0(x - t_4)(x - t_5).$$

3.1 (7 pt) Show that $P_t(x)$ is irreducible in $\mathbb{F}_5[x]$;

3.2 (6 pt) Show that two parameters t, t' give the same polynomial if and only if $t_0 = t'_0$ and $\{t_4, t_5\} = \{t'_4, t'_5\}$.

3.3 (7 pt) Show that every irreducible cubic monic polynomial over $\mathbb{F}_5$ is obtained by this way.

4. (40 pt) For k non-negative integer, let $V_k := \mathbb{R}[x]_{\leq k}$ be the vector space of real polynomials of degree at most k with an action by $\mathrm{SL}_2(\mathbb{R})$ by

$$\gamma \cdot P(x) = (cx+d)^k P\left(\frac{ax+b}{cx+d}\right), \qquad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R}).$$

4.1 (20 pt) Show that V_k is an irreducible representation of $\mathrm{SL}_2(\mathbb{R})$;

4.2 (15 pt) For non-negative integers m, n, consider $V_{m,n} := V_m \otimes V_n$ as a subspace of $\mathbb{C}[x, y]$ of polynomials with both x, y-degrees at most k, with diagonal action of $\mathrm{SL}_2(\mathbb{R})$. Assume $m \geq n \geq 1$. Show that following exact sequence is exact and split as representations of $\mathrm{SL}_2(\mathbb{R})$.

$$0 \longrightarrow V_{m-1,n-1} \xrightarrow{\cdot(y-x)} V_{m,n} \xrightarrow{y=x} V_{m+n} \longrightarrow 0.$$

This implies the following decomposition of representations:

$$V_m \otimes V_n = \oplus_{i=0}^{n} V_{m+n-2i}.$$

4.3 (5 pt) For non-negative integers $\ell \geq m \geq n$ consider the space of invariants $(V_\ell \otimes V_m \otimes V_n)^{\mathrm{SL}_2(\mathbb{R})}$. Show that this space is either trivial or one-dimensional; it is non-trivial if and only if

$$\ell + m + n \equiv 0 \mod 2, \qquad \ell + m \geq n.$$

5. (60 pt)

5.1 (20pt) Find a polynomial $f(x)$ with integer coefficients which has a root over $\mathbb{F}_p$ for each prime p but has not root over $\mathbb{Q}$.

5.2 (20pt) Can you find f irreducible?

5.3 (20pt) What is the smallest possible degree of f?

S.-T. Yau College Student Mathematics Contests 2013

Applied Math. and Computational Math.

Individual

Please solve as many problems as you can!

1. We consider the wave equation $u_{tt} = \Delta u$ in $\mathbb{R}^3 \times \mathbb{R}_+$.

(a): (5 pts) A right going pulse with speed 1

$$u(x, y, z, t) = 1 \text{ for } t < x < t + 1; \quad u(x, y, z, t) = 0 \text{ else}$$

is clearly a solution to the wave equation. However, it is a discontinuous solution, explain in which sense it is a solution to the equation.

(b): (5 pts) Surprisingly, one can construct smooth progressive wave solutions with speed larger than 1. In astronomy this kind of wave known as superluminal wave. Try a solution of the form

$$u(x, y, z, t) = v(\frac{x - ct}{\sqrt{c^2 - 1}}, y, z), \quad c \in \mathbb{R}^3, \quad |c| > 1.$$

Derive an equation for v and show that there is a nontrivial solution with compact support in (y, z) for any fixed x, t.

(c): (5 pts) For any $R > 0$, $0 < t < R$, show that energy

$$E(t) := \int_{|\vec{x}| \leq R-t} \left(|u_t(\cdot, t)|^2 + |\nabla u(\cdot, t)|^2\right) d\vec{x}$$

is a decreasing function.

(c): (10 pts) Show that smooth superluminal progressive wave solutions of the form

$$u(\vec{x}, t) = v(\vec{x} - \vec{c}t), \vec{c} \in \mathbb{R}^3, \quad |\vec{c}| > 1.$$

cannot have a finite energy.

Hint: Using (c) and look at the energy of the solution in various balls.

2. Finite time extinction and hyper-contractiveity are important properties in modeling of some physical and biology systems. The essence of estimates is given by the following problem for ODE.

Assume $y(t) \geq 0$ is a C^1 function for $t > 0$ satisfying $y'(t) \leq \alpha - \beta y(t)^a$ for $\alpha > 0, \beta > 0$, then

(a) (10 points) For $a > 1$, $y(t)$ has the following hyper-contractive property

$$y(t) \leq (\alpha/\beta)^{1/a} + \left[\frac{1}{\beta(a-1)t}\right]^{\frac{1}{a-1}}, \quad \text{for } t > 0.$$

(b) (2 points) For $a = 1$, $y(t)$ decays exponentially

$$y(t) \le \alpha/\beta + y(0)e^{-\beta t}.$$

(c) (10 points) For $a < 1$, $\alpha = 0$, $y(t)$ has finite time extinction, which means that there exists T_{ext} such that $0 < T_{ext} \le \frac{y^{1-a}(0)}{\beta(1-a)}$ and that $y(t) = 0$ for all $t > T_{ext}$.

3. Consider the speed v of a ball (density ρ, radius R) falling through a viscous fluid (density ρ_f, viscosity μ) with drag coefficient given by Stokes' law $\zeta = 6\pi R\mu$:

$$\frac{4}{3}\pi R^3\rho\frac{dv}{dt} = \frac{4}{3}\pi R^3(\rho - \rho_f)g - \zeta v, \quad v(0) = v_0$$

(a): (5 points) Nondimensionalize the equation by writing, $v(t) = V\tilde{v}(\tilde{t})$ with $t = T\tilde{t}$. Select V, T (characteristic scales known as terminal velocity and settling time respectively) so that all coefficients in the ODE but one are equal to 1. Your equation will have a single dimensionless parameter given by the ratio of the initial speed v_0 to the characteristic speed V.

(b): (2 points) Solve the nondimensional problem for $\tilde{v}(\tilde{t})$.

(c): (8 points) Describe the behavior of the solution if the initial speed v_0 is (i) faster than and (ii) slower than the characteristic speed V. Compute the time to reach $(v_0 + V)/2$.

4. Let

$$V_h = \{v : v|_{I_j} \in P^k(I_j) \qquad 1 \le j \le N\}$$

where

$$I_j = (x_{j-1}, x_j), \qquad 1 \le j \le N$$

with

$$x_j = jh, \qquad h = \frac{1}{N}.$$

Here $P^k(I_j)$ denotes the set of polynomials of degree at most k in the interval I_j.

Recall the L^2 projection of a function $u(x)$ into the space V_h is defined by the unique function $u_h \in V_h$ which satisfies

$$||u - u_h|| \le ||u - v|| \qquad \forall v \in V_h$$

where the norm is the usual L^2 norm. We assume $u(x)$ has at least $k+2$ continuous derivatives.

(1) (5 points) Prove the error estimate

$$||u - u_h|| \le Ch^{k+1}$$

Explain how the constant C depends on the derivatives of $u(x)$.

(2) (10 points) If another function $\varphi(x)$ also has at least $k+2$ continuous derivatives, prove

$$|\int_0^1 (u(x) - u_h(x))\varphi(x)dx| \le Ch^{2k+2}$$

Explain how the constant C depends on the derivatives of $u(x)$ and $\varphi(x)$.

5. (15 points) Let $G(V,E)$ be a simple graph of order n and δ the minimum degree of vertices. Suppose that the degree sum of any pair of nonadjacent vertices is at least n and $F \subset E$ with $|F| \le \lfloor \frac{\delta-2}{2} \rfloor$. Let $G-F$ be the graph obtained from G by deleting the edges in F. Prove that

(1) $G-F$ is connected and

(2) $G-F$ is Hamiltonian.

6. (15 points) Let $(F_n)_n$ be the Fibonacci sequence. Namely, $F_0 = 0, F_1 = 1, \ldots, F_{n+2} = F_{n+1} + F_n$.

Establish a relation between $\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^n$ and F_n and use it to design an efficient algorithm that for a given n computes the n-th Fibonacci number F_n. In particular, it must be *more efficient* than computing F_n in n consecutive steps.

Give an estimate on the number of steps of your algorithm.

Hint: Not that if m is even then

$$\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^m = \left(\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^{m/2} \right)^2$$

and if m is odd then

$$\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^m = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^{m-1} \cdot \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$$

and $m-1$ is even.

S.-T. Yau College Student Mathematics Contests 2013

Probability and Statistics Problems

Individual

Please solve 5 out of the following 6 problems.

Problem 1. Let (X_n) be a sequence of random variables.

(1) Assume that $\sum_{n=0}^{\infty} P(|X_n| > n) < \infty$. Prove that $\limsup_{n\to\infty} \frac{|X_n|}{n} \leq 1$.

(2) Prove that (X_n) converges in probability to 0 if and only if for certain $r > 0$, $E\left[\frac{|X_n|^r}{1+|X_n|^r}\right] \to 0$.

Problem 2. Let X and Y be independent $N(0,1)$ random variables.

(1) Find $E[X+Y|X \geq 0, Y \geq 0]$;

(2) Find the distribution function of $X+Y$ given that $X \geq 0$ and $Y \geq 0$.

(Hint: For b) using the fact that $U = (X+Y)/\sqrt{2}$ and $V = (X-Y)/\sqrt{2}$ are independent and $N(0,1)$ distributed.)

Problem 3. Let $\{X_n\}$ be a sequence of independent and identically distributed continuous real valued random variables, and regard n as time. Let A_n be the following event:

$$A_n = \{X_n = \max\{X_1, X_2, \cdots, X_n\}\}.$$

We say that a maximum record occurs at n in such an event.

(1) Evaluate the probability $P(A_n)$.

(2) Denote by Y_n the number of maximum records occurred until time n, i.e.,

$$Y_n = \text{the number of } \ \{1 \leq k \leq n : X_k = \max\{X_1, X_2, \cdots, X_k\}\}.$$

Evaluate the expectation EY_n and the variance DY_n.

Problem 4. Let $X = (X_1, \cdots, X_n)$ be an iid sample from an exponential density with mean θ. Consider testing $H_0 : \theta = \theta_0$ vs. $H_1 : \theta > \theta_0$. Let $P(X) =$ your p-value for an appropriate test.

(a) What is $E_{\theta_0}(P(X))$? Derive your answer explicitly.

(b) Derive $E_\theta(P(X))$ for $\theta \neq \theta_0$. Specifically, assuming only one sample, i.e. $n = 1$, calculate $E_\theta(P(X))$ as explicitly as possible for $\theta \neq \theta_0$.

(c) When there is only one sample, is $E_\theta(P(X))$ a decreasing function of θ? In general, can you prove your result for an arbitrary MLR family?

Problem 5. Let X_1, X_2 be iid uniform on $\theta - \frac{1}{2}$ to $\theta + \frac{1}{2}$.

(a) Show that for any given $0 < \alpha < 1$, you can find $c > 0$ such that

$$P_\theta\{\bar{X} - c < \theta < \bar{X} + c\} = 1 - \alpha,$$

where $\bar{X}$ is the sample mean.

(b) Show that for ϵ positive and sufficiently small

$$P_\theta\{\bar{X} - c < \theta < \bar{X} + c \mid |X_2 - X_1| \geq 1 - \epsilon\} = 1$$

(c) The statement in (a) is used to assert that $\bar{X} \pm c$ is a $100(1-\alpha)\%$ confidence interval for θ. Does the assertion in (b) contradict this? If your sample observations are $X_1 = 1, X_2 = 2$, would you use the confidence interval in (a)?

Problem 6. Suppose you want to estimate the total number of enemy tanks in a war on the basis of the captured tanks. Assume without loss of generality that the tank serial numbers are $1, 2, \cdots, N$, where N is the unknown total number of enemy tanks. Also assume the serial numbers of the n captured tanks are iid uniform on $1, 2, \cdots, N$. (This is a simplified assumption which provides a good approximation if $n << N$).

(a) Find the complete sufficient statistic.

(b) Suggest how you may find the minimum variance unbiased estimate of N.

S.-T. Yau College Student Mathematics Contests 2013

Analysis and Differential Equations

Team

Please solve 5 out of the following 6 problems.

1. Suppose $\Delta = \{z \in \mathbf{C} \mid |z| < 1\}$ is the open unit disk in the complex plane. Show that for any holomorphic function $f : \Delta \to \Delta$,

$$\frac{|f'(z)|}{1 - |f(z)|^2} \leq \frac{1}{1 - |z|^2} \tag{1}$$

for all z in Δ. If equality holds in (1) for some $z_0 \in \Delta$, show that $f \in \mathrm{Aut}(\Delta)$, and that

$$\frac{|f'(z)|}{1 - |f(z)|^2} = \frac{1}{1 - |z|^2}$$

for all $z \in \Delta$.

2. Let f be a function of bounded variation on $[a, b]$, f_1 its generalized derivative as a measure, i.e. $f(x) - f(a) = \int_a^x f_1(y)dy$ for every $x \in [a, b]$ and $f_1(x)$ is an integrable function on $[a, b]$. Let f' be its weak derivative as a generalized function, i.e. $\int_a^b f(x)g'(x)dx = -\int_a^b f'(x)g(x)dx$, for any smooth function $g(x)$ on $[a, b]$, $g(a) = g(b) = 0$. Show that:

a) If f is absolutely continuous, then $f' = f_1$.

b) If the weak derivative f' of f is an integrable function on $[a, b]$, then $f(x)$ is equal to an absolutely continuous function outside a set of measure zero.

3. Show that the convex hull of the roots of any polynomial contains all its critical points as well as all the zeros of higher derivatives of the polynomial. Here the convex hull of a given bounded set in the plane is the smallest convex set containing the given set in the plane.

4. Let $D \subset \mathbf{R}^3$ be an open domain. Show that every smooth vector field $\mathbf{F} = (P, Q, R)$ over D can be written as $\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2$ such that $rot(\mathbf{F}_1) = 0, div(\mathbf{F}_2) = 0$, where $rot(\mathbf{F}) = (\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}, \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}), div(\mathbf{F}) = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$.

5. Let $\mathbf{H}$ be a Hilbert space and $\mathbf{A}$ a compact self-adjoint linear operator over $\mathbf{H}$. Show that there exists an orthor-normal basis of $\mathbf{H}$ consisting of eigenvectors φ_n of $\mathbf{A}$ with non-zero eigenvalues λ_n such that every vector $\xi \in \mathbf{H}$ can be written as: $\xi = \Sigma_k c_k \varphi_k + \xi'$, where $\xi' \in Ker\mathbf{A}, i.e., \mathbf{A}\xi' = 0$. We also have $\mathbf{A}\xi = \Sigma_k \lambda_k c_k \varphi_k$.

If there are infinitely many eigenvectors then $\lim_{n\to\infty} \lambda_n = 0$.

6. A function $f\colon \mathbb{R} \to \mathbb{R}$ is called convex if

$$f(\lambda x + (1-\lambda)x') \leq \lambda f(x) + (1-\lambda) f(x')$$

for $0 \leq \lambda \leq 1$ and each $x, x' \in \mathbb{R}$, and is called strictly convex if

$$f(\lambda x + (1-\lambda)x') < \lambda f(x) + (1-\lambda) f(x')$$

for $0 < \lambda < 1$. We assume that $|f(x)| < \infty$ whenever $|x| < \infty$.

(a) Show that a convex function f is continuous and the function

$$g(y) = \max_{x \in \mathbf{R}} (xy - f(x))$$

is a well-defined convex function over $\mathbf{R}$.

(b) Show that a convex function f is differentiable except at most countably many points.

(c) f is differentiable everywhere if both f and g are strictly convex.

S.-T. Yau College Student Mathematics Contests 2013

Geometry and Topology

Team

Please solve 5 out of the following 6 problems.

1. Let X be the space

$$\{(x,y,0) \mid x^2+y^2=1\} \cup \{(x,0,z) \mid x^2+z^2=1\}$$

Find the fundamental group $\pi_1(\mathbb{R}^3 \setminus X)$.

2. Let M be a smooth connected manifold and $f: M \to M$ be an injective smooth map such that $f \circ f = f$. Show that the image set $f(M)$ is a smooth submanifold in M.

3. Let $T^2 = \{(z,w) \in \mathbb{C}^2 \mid \ |z|=1, |w|=1\}$ be the torus. Define a map $f: T^2 \to T^2$ by $f(z,w) = (zw^3, w)$. Prove that f is a diffeomorphism.

4. Prove: Any 3-dimensional Einstein manifold has constant curvature.

5. State and prove the Myers theorem for complete Riemannian manifolds.

6. Let C be a regular closed curve in $\mathbb{R}^3$. Its torsion is τ. The integral $\frac{1}{2\pi}\int_C \tau ds$ is called the total torsion of C, where s is the arc length parameter. Prove: Given a smooth surface M in $\mathbb{R}^3$, if for any regular closed curve C on M, the total torsion of C is always an integer, then M is a part of a sphere or a plane.

S.-T. Yau College Student Mathematics Contests 2013

Algebra and Number Theory

Team

The exam contains 6 problems. Please choose 5 of them to work on.

1. (20pt) Let A be an $n \times n$ skew symmetric real matrix.

1.1 (10 pt) Prove that all eigenvalues of A are imaginary or zero and that e^A is orthogonal.

1.2 (10 pt) Find conditions on an orthogonal B such that $B = e^A$ is solvable for some skew symmetric and real matrix A.

2. (20pt) Let E/F be a field extension. Let A be an $m \times m$ matrix with entries in E such that $tr(A^n)$ belongs to F for every $n \geq 2$. Show that $tr(A)$ belongs to F by following steps.

2.1 (5pt) Show that there is a polynomial $P(x) = \sum_i a_i x^i \in \bar{E}[x]$ with $a_0 = 1$ such that

$$\sum_i a_i tr(A^{i+k}) = 0, \qquad \forall k \geq 1.$$

2.2 (5pt) Show that we have a polynomial $Q = \sum_i b_i x^i \in F[x]$ with $b_0 = 1$ such that

$$\sum_i b_i tr(A^{i+k}) = 0, \qquad \forall k \geq 2.$$

2.3 (5pt) Let $t \in \bar{E}$ be an eigenvalue of A with multiplicity m invertible in F. Show that $Q(t) = 0$.

2.4 (5pt) Show that $tr(A)$ belongs to F.

Hint: Let $t_i \in \bar{E}$ be all distinct non-zero eigen values of A with multiplicity m_i invertible in F. Then

$$tr(A^n) = \sum_i m_i t_i^n.$$

3. (20pt) Let p be a prime and $G = \mathrm{SL}_2(\mathbb{F}_p)$.

3.1 (10pt) Find the order of G.

3.2 (10pt) Show that the order of every element of G divides either $(p^2 - 1)$ or $2p$.

4. (20pt) Let S_4 be the symmetric group of 4 letters.

4.1 (10pt) Classify all complex irreducible representations of S_4;

4.2 (10pt) Find the character table of S_4.

5. (20pt) Let $\mathbb{F}_2$ be the finite field of two elements.

5.1 (10pt) Find all irreducible polynomials of degree 2 and 3 over $\mathbb{F}_2$;

5.2 (10pt) What is the number of irreducible polynomials of degree 6 over $\mathbb{F}_2$?

6. (20pt) Let F be the splitting field of $x^4 - 2$.

6.1 (10pt) Describe the field F and the Galois group $G = \mathrm{Gal}(F/\mathbb{Q})$.

6.2 (10pt) Describe all subfields K of F and corresponding Galois subgroups $G_K = \mathrm{Gal}(F/K)$.

S.-T. Yau College Student Mathematics Contests 2013

Applied Math. and Computational Math.

Team

Please solve as many problems as you can!

1. Scaling behavior is one of the most important phenomena in scientific modeling and mathematical analysis. The following problem shows the universality and rigidity of scaling limits.

(a): (10 points) Suppose $U > 0$ is an increasing function on $[0, \infty)$ and there is a function $0 < \psi(x) < \infty$ for $x > 0$ such that

$$\lim_{t\to\infty} \frac{U(tx)}{U(t)} = \psi(x), \quad \text{for all } x > 0$$

Then $\psi(x) = x^\alpha$ for some $\alpha \geq 0$.

(b): (10 points) The above problem can be generalized as:

Suppose $U > 0$ is an increasing function on $[0, \infty)$ and there is an extended function $0 \leq \psi(x) \leq \infty$ and a set A dense in $[0, \infty)$ such that

$$\lim_{t\to\infty} \frac{U(tx)}{U(t)} = \psi(x), \quad \text{for all } x \in A$$

Then $\psi(x) = x^\alpha$ for some $\alpha \in [0, \infty]$.

(c): (15 points) (Warming: this part is hard).

A function $L : (0, \infty) \to (0, \infty)$ is called slowly varying at ∞ if

$$\lim_{t\to\infty} \frac{L(tx)}{L(t)} = 1, \quad \text{for all } x \in A \text{ dense in } (0, \infty)$$

The function U in **(a)** and **(b)** can be recast as $U(x) = c\,x^\alpha L(x)$ for some $c \geq 0$. Now we can extend **(b)** to an even more general setting:

Suppose $U > 0$ is increasing on $(0, \infty)$, set A dense in $[0, \infty)$ and

$$\lim_{n\to\infty} a_n U(b_n x) = \psi(x) \leq \infty, \text{ for all } x \in A.$$

where $b_n \to \infty$ and $\frac{a_{n+1}}{a_n} \to 1$ for some interval. Then there is a real number $\alpha \in [0, \infty]$, constant $c \geq 0$, and a function L slowly varying at ∞ such that $\psi(x) = x^\alpha$ and $U(x) = c\,x^\alpha L(x)$.

2. The following three operators are important for many mathematics and physics problems. Let $\phi(x)$ be a smooth periodic function in $\mathbb{T}^n$, Δ, ∇, $\nabla\cdot$ be the standard Laplacian, gradient and divergence operators.

(i): Fokker-Planck operator: $\mathcal{F}u = -\Delta u - \nabla \cdot (u \nabla \phi)$

(ii): Witten Laplacian operator: $\mathcal{W}u = -\Delta u + \nabla\phi \cdot \nabla u$
(iii): Schrödinger operator: $\mathcal{S}u = -\Delta u + \left(\frac{1}{4}|\nabla\phi|^2 - \frac{1}{2}\Delta\phi\right) u$

Show that

(a): (5 points) The Fokker-Planck operator can be recast as $\mathcal{F}u = -\nabla \cdot (e^{-\phi}\nabla(e^{\phi}u)$.
(b): (10 points) These three operators have same eigenvalues.
(c): (5 points) Find all equilibrium solutions for these three operators.

3. (15 points) Let $f(x)$ defined on [0, 1] be a smooth function with sufficiently many derivatives. $x_i = ih$, where $h = \frac{1}{N}$ and $i = 0, 1, \cdots, N$ are uniformly distributed points in [0, 1]. What is the highest integer k such that the numerical integration formula

$$I_N = \frac{1}{N}\left(a_0(f(x_0) + f(x_N)) + a_1(f(x_1) + f(x_{N-1})) + \sum_{i=2}^{N-2} f(x_i)\right)$$

is k-th order accurate, namely

$$\left| I_N - \int_0^1 f(x)dx \right| \leq Ch^k$$

for a constant C independent of h? Please describe the procedure to obtain the two constants a_0 and a_1 for this k.

4. The wave guide problem is defined as

$$u_t + u_x = 0, \qquad v_t - v_x = 0$$

with the boundary condition

$$u(-1, t) = v(-1, t), \qquad v(1, t) = u(1, t)$$

and the initial condition

$$u(x, 0) = f(x), \qquad v(x, 0) = g(x).$$

The upwind scheme for the guide problem is defined as

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} + \frac{u_j^n - u_{j-1}^n}{\Delta x} = 0, \qquad j = -N+1, \cdots, N;$$

$$\frac{v_j^{n+1} - v_j^n}{\Delta t} - \frac{v_{j+1}^n - v_j^n}{\Delta x} = 0, \qquad j = -N, \cdots, N-1;$$

with the boundary condition

$$u_{-N}^{n+1} = v_{-N}^{n+1}, \qquad v_N^{n+1} = u_N^{n+1}$$

where u_j^n and v_j^n approximate $u(x_j, t^n)$ and $v(x_j, t^n)$ respectively at the grid point (x_j, t^n), with $x_j = j\Delta x$, $t^n = n\Delta t$, $\Delta x = \frac{1}{N}$.

(1) (5 points) For the solution to the wave guide problem with the above boundary condition, prove the energy conservation

$$\frac{d}{dt}\int_{-1}^{1}(u^2+v^2)dx = 0.$$

(2) (5 points) For the numerical solution of the the upwind scheme, if we define the discrete energy as

$$E^n = \sum_{j=-N+1}^{N}(u_j^n)^2 + \sum_{j=-N}^{N-1}(v_j^n)^2,$$

prove the discrete energy stability

$$E^{n+1} \le E^n$$

under a suitable time step restriction $\frac{\Delta t}{\Delta x} \le \lambda_0$. You should first find λ_0.

(3) (10 points) Under the same time step restriction, is the numerical solution stable in the maximum norm? That is, can you prove

$$\max_{-N\le j\le N}\max(|u_j^{n+1}|,|v_j^{n+1}|) \le \max_{-N\le j\le N}\max(|u_j^n|,|v_j^n|)?$$

5. (15 points) Let $G=(V,E)$ be a graph of order n. Let X_1, X_2, $\ldots$, X_q with $2\le q\le \kappa(X)$ be subsets of the vertex set V such that $X = X_1\cup X_2\cup\ldots\cup X_q$. If for each i, $i=1,2,\ldots,q$, and for any pair of nonadjacent vertices $x,y\in X_i$, we have

$$d(x)+d(y)\ge n,$$

then X is cyclable in G (*i.e.*, there is a cycle containing all vertices of X.).

Where $d(x)$ is the degree of x and $\kappa(X)$ is the smallest number of vertices separating two vertices of X if X does not induce a complete subgraph of G, otherwise we put $\kappa(X)=|X|-1$ if $|X|\ge 2$ and $\kappa(X)=1$ if $|X|=1$.

6. (15 points) Let $(F_n)_n$ be the Fibonacci sequence. Namely, $F_0 = 0, F_1 = 1, \ldots, F_{n+2} = F_{n+1}+F_n$.

Establish a relation between $\begin{pmatrix}0 & 1\\ 1 & 1\end{pmatrix}^n$ and F_n and use it to design an efficient algorithm that for a given n computes the n-th Fibonacci number F_n. In particular, it must be *more efficient* than computing F_n in n consecutive steps.

Give an estimate on the number of steps of your algorithm.

Hint: Not that if m is even then

$$\begin{pmatrix}0 & 1\\ 1 & 1\end{pmatrix}^m = \left(\begin{pmatrix}0 & 1\\ 1 & 1\end{pmatrix}^{m/2}\right)^2$$

and if m is odd then

$$\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^{m} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^{m-1} \cdot \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$$

and $m-1$ is even.

S.-T. Yau College Student Mathematics Contests 2013

Probability and Statistics Problems

Team

Please solve 5 out of the following 6 problems.

Problem 1. The characteristic function f of a probability distribution function F is defined by

$$f(t) = \int_{-\infty}^{\infty} e^{itx}\, dF(x).$$

Show that $f_1(t) = (\cos t)^2$ is a characteristic function and $f_2(t) = |\cos t|$ is not a characteristic function.

Problem 2. Let $I = [0,1]$ be the unit interval and $\mathcal{B}$ the σ-algebra of Borel sets on I. Let P be the Lebesgue measure on I. Show that on the probability space $(I, \mathcal{B}, P)$ the set of points of x with the following property has probability 1: for all but finitely many rational numbers $p/q \in (0,1)$,

$$\left| x - \frac{p}{q} \right| \geq \frac{1}{(q \log q)^2}.$$

Problem 3. Let X be an integrable random variable, $\mathcal{G}$ a $\sigma-$algebra, and $Y = E[X|\mathcal{G}]$. Assume that X and Y have the same distribution.

(1) Prove that if X is square-integrable, then $X = Y$, a.s. (i.e. X must be $\mathcal{G}$ measurable) ;

(2) Using a) to prove that for any pair of real numbers a, b with $a < b$, we have $\min\{\max\{X, a\}, b\} = \min\{\max\{Y, a\}, b\}$, and consequently, $X = Y$, a.s.

Problem 4. Let $X_1, \cdots, X_n$ be iid $N(\theta, \sigma^2)$, σ^2 known, and let θ have a double exponential distribution, that is, $\pi(\theta) = e^{-|\theta|/a}/(2a)$, a known. A Bayesian test of the hypothesis $H_0 : \theta \leq 0$ versus $H_1 : \theta > 0$ will decide in favor of H_1 if its posterior probability is large.

(a) For a given constant K, calculate the posterior probability that $\theta > K$, that is, $P(\theta > K \mid x_1, \cdots, x_n, a)$.

(b) Find an expression for $\lim_{a\to\infty} P(\theta > K \mid x_1, \cdots, x_n, a)$.

(c) Compare your answer in part (b) to the p-value associated with the classical hypothesis test.

Problem 5. Two sets of interesting ideas emerging in the 1990's are the proposal of model selection with L^1-penalty (e.g., lasso) and the proposal of soft thresholding in simultaneous inferences. Consider a linear model

$$Y = X\beta + \varepsilon,$$

where the set up is as usual (i.e., X is a non-random n by p matrix with $1 < p < n$, $\varepsilon \sim N(0, \sigma^2 \cdot I_n)$ with I_n being the identity matrix). The lasso procedure is to obtain an estimate of the parameter vector β through minimizing

$$(L): \qquad \frac{1}{2}||Y - X\beta||_2^2 + \lambda \cdot ||\beta||_1,$$

which we denote by β^*_λ; here $\lambda > 0$ is a tuning parameter, $||\cdot||_2$ denote the usual L^2 vector norm, and $||\cdot||_1$ denotes the usual L^1 vector norm.

Denote the ordinary least square estimate of β by $\hat{\beta}$, we have

$$\frac{1}{2}||Y - X\beta||_2^2 + \lambda \cdot ||\beta||_1 = \frac{1}{2}||Y - X\hat{\beta}||_2^2 + \frac{1}{2}||X(\beta - \hat{\beta})||_2^2 + \lambda \cdot |\beta||_1. \qquad (0.1)$$

Furthermore, if X has orthonormal columns, e.g.,

$$X'X = I_p,$$

then it can be shown that

$$\beta^*_{\lambda,i} = \begin{cases} \hat{\beta}_i - \gamma, & \hat{\beta}_i > \gamma, \\ 0, & |\hat{\beta}_i| \le \gamma, \\ \hat{\beta}_i + \gamma, & \beta_i < -\gamma. \end{cases} \qquad (0.2)$$

(0.2) is called the *soft thresholding* of $\hat{\beta}_i$'s. This says that with orthonormal design, lasso solution is equivalent to applying soft thresholding to the ordinary least square solution.

(a) Prove equation (0.1) without assuming X is orthogonal.

(b) Show that the lasso estimator is obtained by (0.2) under the assumption that X is orthogonal, and find the relationship between λ and γ.

Problem 6. Consider a usual linear model $Y = X\beta + \varepsilon$, where $\varepsilon \sim N(0, \sigma^2 \cdot I_n)$ and X has n rows and p columns where $1 < p < n$. Consider a p-dimensional column vector $a \neq 0$.

(a) Show that, if $Xa = 0$, then $a'\beta$ is not estimable.

(b) Prove or disprove that, if $a'\beta$ is not estimable, then $Xa = 0$.

(c) Show that X is full rank if and only if $a'\beta$ are estimable for all a.

S.-T. Yau College Student Mathematics Contests 2014

Analysis and Differential Equations

Individual

Please solve 5 out of the following 6 problems.

1. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function which satisfies

$$\sup_{x,y\in\mathbb{R}} |f(x+y) - f(x) - f(y)| < \infty.$$

If we have $\displaystyle\lim_{n\to\infty, n\in\mathbb{N}} \frac{f(n)}{n} = 2014$, prove $\displaystyle\sup_{x\in\mathbb{R}} |f(x) - 2014\,x| < \infty$.

2. Let $f_1, ..., f_n$ are analytic functions on $D = \{z||z| < 1\}$ and continuous on $\bar{D}$, prove that $\phi(z) = |f_1(z)| + |f_2(z)| + ... + |f_n(z)|$ achieves maximum values at the boundary ∂D.

3. Prove that if there is a conformal mapping between the annulus $\{z|r_1 < |z| < r_2\}$ and the annulus $\{z|\rho_1 < |z| < \rho_2\}$, then $\frac{r_2}{r_1} = \frac{\rho_2}{\rho_1}$.

4. Let $U(\xi)$ be a bounded function on $\mathbb{R}$ with finitely many points of discontinuity, prove that

$$P_U(x) = \frac{1}{\pi}\int_{\mathbb{R}} \frac{y}{(x-\xi)^2 + y^2} U(\xi)d\xi$$

is a harmonic function on the upper half plane $\{z \in \mathbb{C}|Imz > 0\}$ and it converges to $U(\xi)$ as $z \to \xi$ at a point ξ where $U(\xi)$ is continuous.

5. Let $f \in L^2(\mathbb{R})$ and let $\hat{f}$ be its Fourier transform. Prove that

$$\int_{-\infty}^{\infty} x^2|f(x)|^2dx \int_{-\infty}^{\infty} \xi^2|\hat{f}(\xi)|^2d\xi \ge \frac{(\int_{-\infty}^{\infty}|f(x)|^2dx)^2}{16\pi^2},$$

under the condition that the two integrals on the left are bounded.

(Hint: Operators $f(x) \to xf(x)$ and $\hat{f}(\xi) \to \xi\hat{f}(\xi)$ after Fourier transform are non-commuting operators. The inequality is a version of the uncertainty principle.)

6. Let Ω be an open domain in the complex plane $\mathbb{C}$. Let $\mathbb{H}$ be the subspace of $L^2(\Omega)$ consisting of holomorphic functions on Ω.

a) Show that $\mathbb{H}$ is a closed subspace of $L^2(\Omega)$, and hence is a Hilbert space with inner product

$$(f, g) = \int_{\Omega} f(z)\bar{g}(z)dxdy, \text{ where } z = x + iy.$$

b) If $\{\phi_n\}_{n=0}^{\infty}$ is an orthonormal basis of $\mathbb{H}$, then

$$\Sigma_{n=0}^{\infty}|\phi_n(z)|^2 \leq \frac{c^2}{d(z,\Omega^c)}, \text{ for } z \in \Omega.$$

c) The sum

$$B(z,w) = \Sigma_{n=0}^{\infty}\phi_n(z)\bar{\phi}_n(w)$$

converges absolutely for $(z,w) \in \Omega \times \Omega$, and is independent of the choice of the orthonormal basis.

S.-T. Yau College Student Mathematics Contests 2014

Probability and Statistics Problems

Individual

Please solve the following 5 problems.

Problem 1. Let X be a real valued random variable such that for all smooth functions $f : R \to R$ with compact support we have $E[Xf(X)] = E[f'(X)]$. Show that X has the standard normal distribution.

Problem 2. Let (X_n) be a sequence of uncorrelated random variables of mean zero such that

$$\sum_{n=1}^{\infty} nE|X_n|^2 < \infty.$$

Show that $S_n = \sum_{i=1}^{n} X_i$ converges almost surely.

Problem 3. Let $(\Omega, \mathcal{F})$ be a measurable space and $\mathcal{G}$ be a sub-σ-field of $\mathcal{F}$. Let P and Q be two probabilities which are mutually absolutely continuous on $\mathcal{F}$. We denote by X_0 the Radon-Nikodym density of Q with respect to P on $\mathcal{F}$. Show that the following two properties are satisfied:

(a) $0 < E_P[X_0|\mathcal{G}] < +\infty$, P-a.s.;

(b) for every $\mathcal{F}$-measurable non-negative random variable f,

$$E_P[fX_0|\mathcal{G}] = E_Q[f|\mathcal{G}]E_P[X_0|\mathcal{G}].$$

Problem 4. Suppose $X_1, \ldots, X_n, \ldots$ is a sequence of random numbers drawn from the uniform distribution $U(0,1)$. One observes these numbers sequentially. At time n, one keeps a record of $Y_n \stackrel{def}{=} X_{(n)} = \max_{i=1}^{n} X_i = \max\{Y_{n-1}, X_n\}$ and $Z_n \stackrel{def}{=} \bar{X}_n = \sum_{i=1}^{n} X_i/n = (n-1)/nZ_{n-1} + 1/nX_n$ and discards all previous recordings.

(a) What is the best guess of X_1 if one only observes Y_n?

(b) What is the best guess of X_1 if one only observes Z_n?

(c) Comparing the two guesses of X_1, which one is better (and in what sense)?

Give good reasoning to justify your answers.

Problem 5. Suppose we take a random sample of size n from a bag of colored balls (red, blue and yellow balls) with replacement. Let X_1 denote the number of red balls, X_2 denote the number of blue balls, and X_3 denote the number of yellow balls in the sample. Assuming we know that the total number of yellow balls is triple the total number of red balls in the bag. Or in other words, the red, blue and yellow balls occur with probability p_1, p_2 and $p_3 = 3p_1$, respectively in the bag.

1. Find the aymptotic distribution (after appropriate normalization) for the MLE of p_2.

2. Construct the likelihood ratio test statistic for the null hypothesis that $p_1 = p_2 = p_3/3$ (the alternative is that $p_1 = p_2 = p_3/3$ is not true). What is the asymptotic distribution of your test statistic under null?

S.-T. Yau College Student Mathematics Contests 2014

Geometry and Topology

Individual

Please solve 5 out of the following 6 problems.

1. Let X be the quotient space of S^2 under the identifications $x \sim -x$ for x in the equator S^1. Compute the homology groups $H_n(X)$. Do the same for S^3 with antipodal points of the equator $S^2 \subset S^3$ identified.

2. Let $M \to \mathbb{R}^3$ be a graph defined by $z = f(u, v)$ where $\{u, v, z\}$ is a Descartes coordinate system in $\mathbb{R}^3$. Suppose that M is a minimal surface. Prove that:
(a) The Gauss curvature K of M can be expressed as

$$K = \Delta \log \left(1 + \frac{1}{W}\right), \quad W := \sqrt{1 + \left(\frac{\partial f}{\partial u}\right)^2 + \left(\frac{\partial f}{\partial v}\right)^2},$$

where Δ denotes the Laplacian with respect to the induce metric on M (i.e., the first fundamental form of M).
(b) If f is defined on the whole uv-plane, then f is a linear function (Bernstein theorem).

3. Let $M = \mathbb{R}^2/\mathbb{Z}^2$ be the two dimensional torus, L the line $3x = 7y$ in $\mathbb{R}^2$, and $S = \pi(L) \subset M$ where $\pi : \mathbb{R}^2 \to M$ is the projection map. Find a differential form on M which represents the Poincaré dual of S.

4. Let $p : (\tilde{M}, \tilde{g}) \to (M, g)$ be a Riemannian submersion. This is a submersion $p : \tilde{M} \to M$ such that for each $x \in \tilde{M}$, $Dp : \ker^{\perp}(Dp) \to T_{p(x)}(M)$ is a linear isometry.

(a) Show that p shortens distances.
(b) If $(\tilde{M}, \tilde{g})$ is complete, so is (M, g).
(c) Show by example that if (M, g) is complete, $(\tilde{M}, \tilde{g})$ may not be complete.

5. Let $\Psi : M \to \mathbb{R}^3$ be an isometric immersion of a compact surface M into $\mathbb{R}^3$. Prove that $\int_M H^2 d\sigma \geq 4\pi$, where H is the mean curvature of M and $d\sigma$ is the area element of M.

6. The unit tangent bundle of S^2 is the subset

$$T^1(S^2) = \{(p, v) \in \mathbb{R}^3 \mid \|p\| = 1,\ (p, v) = 0 \text{ and } \|v\| = 1\}.$$

Show that it is a smooth submanifold of the tangent bundle $T(S^2)$ of S^2 and $T^1(S^2)$ is diffeomorphic to $\mathbb{R}P^3$.

S.-T. Yau College Student Mathematics Contests 2014

Applied Math. and Computational Math.

Individual

Please solve as many problems as you can!

1. (20 pts) Ming Antu (1692-1763) is one of the greatest Chinese/Mongolian mathematicians. In the 1730s, he first established and used what was later to be known as Catalan numbers (Euler (1707-1763) rediscovered them around 1756; Belgian mathematician Eugene Catalan (1814-1894) "rediscovered" them again in 1838),

$$c_n = \frac{1}{n+1}\binom{2n}{n}, \quad n = 0, 1, 2, \cdots$$

and Ming Antu derived the following half-angle formula in 1730:

$$\sin^2\frac{\theta}{2} = \sum_{n=1}^{\infty} c_{n-1}\left(\frac{\sin\theta}{2}\right)^{2n}$$

Prove this formula.

Hint: you may use generating function

$$F(z) = \sum_{n=0}^{\infty} c_n z^n$$

and show that $\sum_{m+k=n} c_m c_k = c_{n+1}$ *and then show* $zF(z)^2 = F(z) - 1$.

2. *Many algorithms, including polynomial factorisation in finite fields, require to compute* $\gcd(f(X), X^N - 1)$ *for a polynomial* f *of reasonably small degree* n *and a binomial* $X^N - 1$ *of very large degree* N. *Since* N *is very large the direct application of the Euclid algorithm is very inefficient.*

Questions:

(i) (10 pts) Suggest a more efficient approach the direct computation of $\gcd(f(X), X^N - 1)$ via the Euclid algorithm.
(ii) (10 pts) Generalise it to $\gcd(f(X), A_1X^{N_1} + \ldots + A_mX^{N_m} + A_{m+1})$.

Hint: If for three polynomials f, g *and* h *we have* $g \equiv h \pmod f$ *then*

$$\gcd(f, g) = \gcd(f, h).$$

3. For solving the following partial differential equation

$$u_t + f(u)_x = 0, \qquad 0 \le x \le 1 \tag{1}$$

where $f'(u) \ge 0$, with periodic boundary condition, we can use the following semi-discrete upwind scheme

$$\frac{d}{dt}u_j + \frac{f(u_j) - f(u_{j-1})}{\Delta x} = 0, \qquad j = 1, 2, \cdots, N, \tag{2}$$

with periodic boundary condition

$$u_0 = u_N, \tag{3}$$

where $u_j = u_j(t)$ approximates $u(x_j, t)$ at the grid point $x = x_j = j\Delta x$, with $\Delta x = \frac{1}{N}$.

(i) (15 pts) Prove the following L^2 stability of the scheme

$$\frac{d}{dt}E(t) \le 0 \tag{4}$$

where $E(t) = \sum_{j=1}^N |u_j|^2 \Delta x$.

(ii) (15 pts) Do you believe (4) is true for $E(t) = \sum_{j=1}^N |u_j|^{2p} \Delta x$ for *arbitrary* integer $p \ge 1$? If yes, prove the result. If not, give a counter example.

4. Let A be an $n \times n$ matrix with real and positive eigenvalues and b be a given vector. Consider the solution of $Ax = b$ by the following Richardson's iteration

$$x^{(k+1)} = (I - \omega A)x^{(k)} + \omega b$$

where ω is a damping coefficient. Let λ_1 and λ_n be the smallest and the largest eigenvalues of A. Let $G_\omega = I - \omega A$.

(i) (4 points) Prove that the Richardson's iteration converges if and only if

$$0 < \omega < \frac{2}{\lambda_n}.$$

(ii) (8 points) Prove that the optimal choice of ω is given by

$$\omega_{\text{opt}} = \frac{2}{\lambda_1 + \lambda_n}.$$

Prove also that

$$\rho(G_\omega) = \begin{cases} 1 - \omega\lambda_1 & \omega \le \omega_{\text{opt}} \\ (\lambda_n - \lambda_1)/(\lambda_n + \lambda_1) & \omega = \omega_{\text{opt}} \\ \omega\lambda_n - 1 & \omega \ge \omega_{\text{opt}} \end{cases}$$

where $\rho(G_\omega)$ is the spectral radius of G_ω.

(iii) (8 points) Prove that, if A is symmetric and positive definite, then

$$\rho(G_{\omega_{\text{opt}}}) = \frac{\kappa_2(A) - 1}{\kappa_2(A) + 1}$$

where $\kappa_2(A)$ is the spectral condition number of A.

5. (10 pts) For solving the following heat equation on interval

$$u_t = u_{xx}, \qquad 0 \le x \le 1 \tag{5}$$

with boundary condition

$$u(0) = u_0, \quad u(1) = u_1, \tag{6}$$

we first discretize the interval $[0, 1]$ into N subintervals uniformly, that is, the mesh size $h = 1/N$. We choose a temporal step size k and approximate the solution $u(jh, nk)$ by U_j^n, $j = 1, ..., N-1$, $n = 0, 1, 2, ...$ Using the backward Euler method in time and central finite difference in space, the discrete function U_j^n satisfies:

$$U_j^{n+1} - U_j^n = \lambda(U_{j-1}^{n+1} - 2U_j^{n+1} + U_{j+1}^{n+1}), \quad j = 1, ..., N-1, \tag{7}$$

where $\lambda = k/h^2$, and

$$U_0^{n+1} = u_0, \ U_N^{n+1} = u_1.$$

Show that

$$\frac{1}{2}\sum_{j=1}^{N-1}\left((U_j^{n+1})^2 - (U_j^n)^2\right) \le -\lambda \sum_{j=1}^{N-2}(U_{j+1}^{n+1} - U_j^{n+1})^2$$
$$- \frac{\lambda}{2}((U_1^{n+1})^2 + (U_{N-1}^{n+1})^2) + \frac{\lambda}{2}(u_0^2 + u_1^2) \tag{8}$$

S.-T. Yau College Student Mathematics Contests 2014

Algebra and Number Theory

Individual

This exam of 6 problems is designed to test how much you know rather than how much you don't know. You are not expected to finish all problems but do as much as you can.

Problem 1. Let G be a finite subgroup of $\mathrm{GL}(V)$ where V is an n-dimensional complex vector space.

(a) (5 points) Let

$$H = \{h \in G : hv = \eta(h)v \text{ for some } \eta(h) \in \mathbb{C}^\times \text{ and all } v \in V\}.$$

Prove that H is a normal subgroup of G and that the map $h \mapsto \eta(h)$ is an isomorphism between H and its image in $\mathbb{C}^\times$.

(b) (5 points) Let χ_V be the character function of G acting on V, i.e., $\chi_V(g) = \mathrm{tr}(g)$ with g viewed as an automorphism of V. Prove $|\chi_V(g)| \le n$ for all $g \in G$, and the equality holds if and only if $g \in H$.

(c) (10 points) Let W be an irreducible representation of G. Then W is isomorphic to a direct summand of $V^{\otimes m}$ for some m (as representations of G).

Problem 2. Let $a_1, \ldots, a_n$ be nonnegative real numbers.

(a) (6 points) Prove that the $n \times n$ matrix $A = (t^{a_i + a_j})$ is positive semi-definite for every real number $t > 0$. Find the rank of A.

(b) (7 points) Let $B = (c_{ij})_{n \times n}$ be an $n \times n$-matrix with $c_{ij} = \frac{1}{1+a_i+a_j}$. Prove that A is a positive semi-definite matrix.

(c) (7 points) Prove that B is positive definite if and only if a_i are all distinct.

Problem 3. Consider the equations

$$X^2 - 82Y^2 = \pm 2$$

(a) (5 points) Show that if (x, y) is a solution for $X^2 - 82Y^2 = \pm 2$, then $(9x - 82y, x - 9y)$ is a solution for $X^2 - 82Y^2 = \mp 2$.

(b) (7 points) Show that the equations have solutions over $\mathbb{Z}/p^n\mathbb{Z}$ for any n and odd prime p.

(c) (8 points) Show that the equations have no solutions over $\mathbb{Z}$.

Problem 4. Let S and T be nonabelian finite simple groups, and write $G = S \times T$.

(a) (7 points) Show that the total number of normal subgroups of G is four.

(b) (6 points) If S and T are isomorphic, show that G has a maximal proper subgroup not containing either direct factor.

(c) (7 points) If G has a maximal proper subgroup that contains neither of the direct factors of G, show that S and T are isomorphic.

Problem 5. (20 points) Let $\mathbb{F}$ be a finite field and $f_i \in \mathbb{F}[X_1, X_2, ..., X_n]$ be polynomials of degree d_i, where $1 \le i \le r$, such that $f_i(0, ..., 0) = 0$ for all i. Show that if

$$n > \sum_{i=1}^{r} d_i,$$

then there exists nonzero solution to the system of equations: $f_i = 0$, for all $1 \le i \le r$. (Hint: you may first verify that the number of integral solutions is congruent to the following number modulo p

$$\sum_{X \in \mathbb{F}^n} \prod_{i=1}^{r} \left(1 - f_i(X)^{q-1}\right).$$

)

Problem 6.

(a) (5 points) Let A and B be two real $n \times n$ matrices such that $AB = BA$. Show that $\det(A^2 + B^2) \ge 0$.

(b) (15 points) Generalize this to the case of k pairwise commuting matrices.

S.-T. Yau College Student Mathematics Contests 2014

Analysis and Differential Equations

Team

Please solve 5 out of the following 6 problems.

1. Calculate the integral:

$$\int_0^\infty \frac{\log x}{1+x^2} dx.$$

2. Construct an increasing function on $\mathbb{R}$ whose set of discontinuities is precisely $\mathbb{Q}$.

3. Prove that any bounded analytic function F over $\{z|r < |z| < R\}$ can be written as $F(z) = z^\alpha f(z)$, where f is an analytic function over the disk $\{z||z| < R\}$ and α is a constant.

4. Let $D \subset \mathbb{R}^n$ be a bounded open set, $f : \bar{D} \to \bar{D}$ is a smooth map such that its Jacobian $\left|\frac{\partial f}{\partial x}\right| \equiv 1$, where $\bar{D}$ denotes the closure of D. Prove

(a) for each small ball $B_\epsilon(x)$, there exists a positive integer k such that $f^k(B_\epsilon(x)) \cap B_\epsilon(x) \neq \varnothing$, where $B_\epsilon(x)$ denotes the ball centered at x with radius ϵ;

(b) there exists $x \in \bar{D}$ and a sequence $k_1, k_2, \cdots k_j, \cdots$ such that $f^{k_j}(x) \to x$ as $k_j \to \infty$.

5. Let u be a subharmonic function over a domain $\Omega \subset \mathbf{C}$, i.e., it is twice differentiable and $\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \geq 0$. Prove that u achieves its maximum in the interior of Ω only when u is a constant.

6. Suppose that $\phi \in C_0^\infty(\mathbf{R}^n)$, $\int_{\mathbf{R}^n} \phi dx = 1$. Let $\phi_\epsilon(x) = \epsilon^{-n}\phi(x/\epsilon), x \in \mathbf{R}^n, \epsilon > 0$. Prove that if $f \in L^p(\mathbf{R}^n), 1 \leq p < \infty$, then $f * \phi_\epsilon \to f$ in $L^p(\mathbf{R}^n)$, as $\epsilon \to 0$. It is not true for $p = \infty$.

S.-T. Yau College Student Mathematics Contests 2014

Probability and Statistics Problems

Team

Please solve the following 5 problems.

Problem 1. Suppose that X_n converges to X in distribution and Y_n converges to a constant c in distribution. Show that

(a) Y_n converges to c in probability;

(b) X_nY_n converges to cX in distribution.

Problem 2. Let X and Y be two random variables with $|Y| > 0$, a.s.. Let $Z = X/Y$.

(a) Assume the distribution function of (X, Y) has the density $p(x, y)$. What is the density function of Z?

(b) Assume X and Y are independent and X is $N(0, 1)$ distributed, Y has the uniform distribution on $(0, 1)$. Give the density function of Z.

Problem 3. Let $(\Omega, \mathcal{F}, P)$ be a probability space.

(a) Let $\mathcal{G}$ be a sub σ-algebra of $\mathcal{F}$, and $\Gamma \in \mathcal{F}$. Prove that the following properties are equivalent:

(i) Γ is independent of $\mathcal{G}$ under P,

(ii) for every probability Q on $(\Omega, \mathcal{F})$, equivalent to P, with dQ/dP being $\mathcal{G}$ measurable, we have $Q(\Gamma) = P(\Gamma)$.

(b) Let X, Y, Z be random variables and Y is integrable. Show that if (X, Y) and Z are independent, then $E[Y|X, Z] = E[Y|X]$.

Problem 4. Let $X_1, ..., X_n$ be i.i.d. $N(0, \sigma^2)$, and let M be the mean of $|X_1|, ..., |X_n|$.

1. Find $c \in R$ so that $\hat{\sigma} = cM$ is a consistent estimator of σ.
2. Determine the limiting distribution for $\sqrt{n}(\hat{\sigma} - \sigma)$.
3. Identify an approximate $(1 - \alpha)\%$ confidence interval for σ.
4. Is $\hat{\sigma} = cM$ asymptotically efficient? Please justify your answer.

Problem 5. The shifted exponential distribution has the density function

$$f(y; \phi, \theta) = 1/\theta \exp\{-(u - \phi)/\theta\}, \qquad y > \phi, \theta > 0.$$

Let $Y_1, \ldots, Y_n$ be a random sample from this distribution. Find the maximum likelihood estimator (MLE) of ϕ and θ and the limiting distribution of the MLE.
You may use the following Rényi representation of the order statistics: Let $E_1, \ldots, E_n$, be a random sample from the standard exponential distribution (i.e., the above distribution with $\phi = 0, \theta = 1$). Let $E_{(r)}$ denote the r-th order statistics. According to the Rényi representation,

$$E_{(r)} \stackrel{D}{=} \sum_{j=1}^{r} \frac{E_j}{n+1-j}, \qquad r = 1, \ldots, n.$$

Here, the symbol $\stackrel{D}{=}$ denotes equal in distribution.

S.-T. Yau College Student Mathematics Contests 2014

Geometry and Topology

Team

Please solve 5 out of the following 6 problems.

1. Compute the fundamental and homology groups of the wedge sum of a circle S^1 and a torus $T = S^1 \times S^1$.

2. Given a properly discontinuous action $F : G \times M \to M$ on a smooth manifold M, show that M/G is orientable if and only if M is orientable and $F(g, \cdot)$ preserves the orientation of M. Use this statement to show that the Möbius band is not orientable and that $\mathbb{R}P^n$ is orientable if and only if n is odd.

3. (a) Consider the space Y obtained from $S^2 \times [0, 1]$ by identifying $(x, 0)$ with $(-x, 0)$ and also identifying $(x, 1)$ with $(-x, 1)$, for all $x \in S^2$. Show that Y is homeomorphic to the connected sum $\mathbb{R}P^3 \# \mathbb{R}P^3$.

(b) Show that $S^2 \times S^1$ is a double cover of the connected sum $\mathbb{R}P^3 \# \mathbb{R}P^3$.

4. Prove that a bi-invariant metric on a Lie group G has nonnegative sectional curvature.

5. Let M be the upper half-plane $\mathbb{R}^2_+$ with the metric

$$ds^2 = \frac{dx^2 + dy^2}{y^k}.$$

For which values of k is M complete?

6. Given any nonorientable manifold M show the existence of a smooth orientable manifold $\overline{M}$ which is a double covering of M. Find $\overline{M}$ when M is $\mathbb{R}P^2$ or the Möbius band.

S.-T. Yau College Student Mathematics Contests 2014

Algebra and Number Theory

Team

Solve 5 out of 6 problems, or the highest 5 scores will be counted.

Problem 1. Let the special linear group (of order 2)

$$\mathrm{SL}_2(\mathbb{R}) = \{g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{R}) : \det g = 1\}$$

act on the upper half plane $\mathbb{H} = \{z = x + iy \in \mathbb{C} : y > 0\}$ linear fractionally:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} z = \frac{az+b}{cz+d}.$$

(a) (5 points) Prove that the action is transitive, i.e., for any two $z_1, z_2 \in \mathbb{H}$, there is $g \in \mathrm{SL}_2(\mathbb{R})$ such that $gz_1 = z_2$.

(b) (5 points) For a fixed $z \in \mathbb{H}$, prove that its stabilizer $G_z = \{g \in \mathrm{SL}_2(\mathbb{R}) : gz = z\}$ is isomorphic to $\mathrm{SO}_2(\mathbb{R}) = \{g \in M_2(\mathbb{R}) : gg^t = 1\}$, where g^t is the transpose of g.

(c)(10 points) Let $\mathbb{Z}$ be the set of integers and let

$$\Gamma(2) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R}) : a, b, c, d \in \mathbb{Z}, \quad a - 1 \equiv d - 1 \equiv b \equiv c \equiv 0 \pmod 2 \right\}$$

be a discrete subgroup of $\mathrm{SL}_2(\mathbb{R})$ (no need to prove this), and let it act on $\mathbb{Q} \cup \{\infty\}$ linearly fractionally as above. How many orbits does this action have? Give a representative for each orbit.

Problem 2. Let $p \geq 7$ be an odd prime number.

(a) (5 points) (to warm up) Evaluate the rational number $\cos(\pi/7) \cdot \cos(2\pi/7) \cdot \cos(3\pi/7)$.

(b) (15 points) Show that $\prod_{n=1}^{(p-1)/2} \cos(n\pi/p)$ is a rational number and determine its value.

Problem 3. (20 points, 10 points each) For any 3×3 matrix $A \in M_3(\mathbb{Q})$, let A^{db} be the 6×6 matrix

$$A^{db} := \begin{pmatrix} 0 & I_3 \\ A & 0 \end{pmatrix}$$

(a) Express the characteristic and minimal polynomials of A^{db} over $\mathbb{Q}$ in terms of the characteristic and minimal polynomial of A.

(b) Suppose that $A, B \in M_3(\mathbb{Q})$ are such that A^{db} and B^{db} are conjugate in the sense that there exists an element $C \in GL_6(\mathbb{Q})$ such that $C \cdot A^{db} \cdot C^{-1} = B^{db}$. Are A and B conjugate? (Either prove this statement or give a counterexample.)

Problem 4. (20 points) Classify all groups of order 8.

Problem 5. Let V be a finite dimensional vector space over complex field $\mathbb{C}$ with a non-degenerate symmetric bilinear form $(\ ,\ )$. Let

$$O(V) = \{g \in \mathrm{GL}(V) | (gu, gv) = (u, v),\ u, v \in V\}$$

be the orthogonal group.

(a) (10 points) Prove that

$$(V \otimes_{\mathbb{C}} V)^{O(V)} \cong \mathrm{End}_{O(V)}(V),$$

and construct one such isomorphism. Here $O(V)$ acts on $V \otimes_{\mathbb{C}} V$ via $g(a \otimes b) = ga \otimes gb$, and $(V \otimes_{\mathbb{C}} V)^{O(V)}$ is the fixed point subspace of $V \otimes V$.

(b) (10 points) Prove that the fixed point subspace $(V \otimes_{\mathbb{C}} V)^{O(V)}$ is 1-dimensional.

Problem 6. (20 points) Let c be a non-zero rational integer.

(a) (6 points) Factorize the three variable polynomial

$$f(x, y, z) = x^3 + cy^3 + c^2z^3 - 3cxyz$$

over $\mathbb{C}$ (you may assume $c = \theta^3$ for some $\theta \in \mathbb{C}$).

(b) (7 points) When $c = \theta^3$ is a cube for some rational integer θ, prove that there are only finitely many integer solutions $(x, y, z) \in \mathbb{Z}^3$ to the equation $f(x, y, z) = 1$.

(c) (7 points) When c is not a cube of any rational integers, prove that there infinitely many integer solutions $(x, y, z) \in \mathbb{Z}^3$ to the equation $f(x, y, z) = 1$.

S.-T. Yau College Student Mathematics Contests 2014

Applied Math. and Computational Math.

Team

Please solve as many problems as you can!

1. (15 pts)
Given a finite positive (Borel) measure $d\mu$ on $[0,1]$, define its sequence of moments as follows

$$c_j = \int_0^1 x^j \, d\mu(x), \quad j = 0, 1, \ldots.$$

Show that the sequence is *completely monotone* in the sense that that

$$(I-S)^k c_j \geq 0 \quad \text{for all } j, k \geq 0,$$

where S denotes the backshift operator given by $Sc_j = c_{j+1}$ for $j \geq 0$.

2. (20 pts)
We recall that a polynomial

$$f(X) = a_d X^d + a_{d-1} X^{d-1} + \cdots + a_1 X + a_0 \in \mathbb{Z}[X]$$

is called an Eisenstein polynomial if for some prime p we have

(i) $p \mid a_i$ *for* $i = 0, \ldots, d-1$,
(ii) $p^2 \nmid a_0$,
(iii) $p \nmid a_d$.

Eisenstein polynomials are well-know to be irreducible over $\mathbb{Z}$, so they can be used to construct explicit examples of irreducible polynomials.

Questions:

(i) Prove that a composition $f(g(X))$ of two Eisenstein polynomials f and g is an Eisenstein polynomial again.
(ii) Suggest a multivariate generalisation of the Eisenstein polynomials. That is, describe a class polynomials $F(X_1, \ldots, X_m)$ in terms of the divisibility properties of their coefficients that are guaranteed to be irreducible.

3. (20 pts) For solving the following partial differential equation

$$u_t + f(u)_x = 0, \qquad 0 \leq x \leq 1 \tag{1}$$

where $f'(u) \geq 0$, with periodic boundary condition, we can use the following semi-discrete discontinuous Galerkin method: Find $u_h(\cdot, t) \in V_h$ such that, for all $v \in V_h$ and $j = 1, 2, \cdots, N$,

$$\int_{I_j} (u_h)_t v dx - \int_{I_j} f(u_h) v_x dx + f((u_h)^-_{j+1/2}) v^-_{j+1/2} - f((u_h)^-_{j-1/2}) v^+_{j-1/2} = 0, \tag{2}$$

with periodic boundary condition

$$(u_h)^-_{1/2} = (u_h)^-_{N+1/2}; \quad (u_h)^+_{N+1/2} = (u_h)^+_{1/2}, \tag{3}$$

where $I_j = (x_{j-1/2}, x_{j+1/2})$, $0 = x_{1/2} < x_{3/2} < \cdots < x_{N+1/2} = 1$, $h = \max_j(x_{j+1/2} - x_{j-1/2})$, $v^{\pm}_{j+1/2} = v(x^{\pm}_{j+1/2}, t)$, and

$$V_h = \{v : v|_{I_j} \text{ is a polynomial of degree at most } k \text{ for } 1 \le j \le N\}.$$

Prove the following L^2 stability of the scheme

$$\frac{d}{dt}E(t) \le 0 \tag{4}$$

where $E(t) = \int_0^1 (u_h(x,t))^2 dx$.

4. Consider the linear system $Ax = b$. The GMRES method is a projection method which obtains a solution in the m-th Krylov subspace K_m so that the residual is orthogonal to AK_m. Let r_0 be the initial residual and let $v_0 = r_0$. The Arnoldi process is applied to build an orthonormal system $v_1, v_2, \cdots, v_{m-1}$ with $v_1 = Av_0/\|Av_0\|$. The approximate solution is obtained from the following space

$$K_m = \text{span}\{v_0, v_1, \cdots, v_{m-1}\}.$$

(i) (5 points) Show that the approximate solution is obtained as the solution of a least-square problem, and that this problem is triangular.

(ii) (5 points) Prove that the residual r_k is orthogonal to $\{v_1, v_2, \cdots, v_{k-1}\}$.

(iii) (5 points) Find a formula for the residual norm.

(iv) (5 points) Derive the complete algorithm.

5. (10 pts)

(i) Set $x_0 = 0$. Write the recurrence

$$x_k = 2x_{k-1} + b_k, \quad k = 1, 2, \cdots, n,$$

in a matrix form $A\vec{x} = \vec{b}$. For $b_1 = -1/3$, $b_k = (-1)^k, k = 2, 3, \cdots, n$, verify that $x_k = (-1)^k/3, k = 1, 2, \cdots, n$ is the exact solution.

(ii) Find A^{-1} and compute condition number of A in L^1 norm.

S.-T. Yau College Student Mathematics Contests 2015

Analysis and Differential Equations

Individual

Please solve 5 out of the following 6 problems.

1. Let $f_n \in L^2(R)$ be a sequence of measurable functions over the line, $f_n \to f$ almost everywhere. Let $||f_n||_{L^2} \to ||f||_{L^2}$, prove that $||f_n - f||_{L^2} \to 0$.

2. Let f be a continuous function on $[a, b]$, define $M_n = \int_a^b f(x)x^n dx$. Suppose that $M_n = 0$ for all integers $n \geq 0$, show that $f(x) = 0$ for all x.

3. Determine all entire functions f that satisfying the inequality

$$|f(z)| \leq |z|^2 |Im(z)|^2$$

for z sufficiently large.

4. Describe all functions that are holomorphic over the unit disk $D = \{z||z| < 1\}$, continuous on $\bar{D}$ and map the boundary of the disk into the boundary of the disk.

5. Let $T : H_1 \to H_2, Q : H_2 \to H_1$ be bounded linear operators of Hilbert spaces H_1, H_2. Let $QT = Id - S_1, TQ = Id - S_2$ where S_1 and S_2 are compact operators. Prove $KerT = \{v \in H_1, Tv = 0\}, CokerT = H_2/\overline{Im(T)}$ are finite dimensional and $Im(T) = \{Tv \in H_2, v \in H_1\}$ is closed in H_2.

Note: S is compact means for every bounded sequence $x_n \in H_1, Sx_n$ has a converging subsequence.

6. Let H_1 be the Sobolev space on the unit interval $[0, 1]$, i.e. the Hilbert space consisting of functions $f \in L^2([0, 1])$ such that

$$||f||_1^2 = \Sigma_{n=-\infty}^{\infty}(1 + n^2)|\hat{f}(n)|^2 < \infty;$$

where

$$\hat{f}(n) = \frac{1}{2\pi} \int_0^1 f(x)e^{-2\pi inx} dx$$

are Fourier coefficients of f. Show that there exists constant $C > 0$ such that

$$||f||_{L^\infty} \leq C||f||_1$$

for all $f \in H_1$, where $||.||_{L^\infty}$ stands for the usual supremum norm. (Hint: Use Fourier series.)

S.-T. Yau College Student Mathematics Contests 2015

Probability and Statistics

Individual (5 problems)

Problem 1. (a) Let X and Y be two random variables with zero means, variance 1, and correlation ρ. Prove that

$$\mathbb{E}[\max\{X^2, Y^2\}] \leq 1 + \sqrt{1-\rho^2}.$$

(b) Let X and Y have a bivariate normal distribution with zero means, variances σ^2 and τ^2, respectively, and correlation ρ. Find the conditional expectation $\mathbb{E}(X|Y)$.

Problem 2. We flip a fair coin until heads turns out twice consecutively. What is the expected number of flips?

Problem 3. Let $(X_n, n \geq 1)$ be a sequence of independent Gaussian variables, with respective mean μ_n, and variance σ_n^2.

(a) Prove that if $\sum_n X_n^2$ converges in L^1, then $\sum_n X_n^2$ converges in L^p, for every $p \in [1, \infty)$.

(b) Assume that $\mu_n = 0$, for every n. Prove that if $\sum_n \sigma_n^2 = \infty$, then

$$\mathbb{P}(\sum_n X_n^2 = \infty) = 1.$$

Problem 4. Let $X_1, \ldots, X_n$ be a random sample of size n from the exponential distribution with pdf $f(x;\theta) = \theta^{-1}\exp(-x/\theta)$ for $x, \theta > 0$, zero elsewhere. Let $Y_1 = \min\{X_1, \ldots, X_n\}$. Consider an estimator nY_1.

(a) Show this estimate is unbiased.

(b) Prove or disprove: This estimate is a consistent estimator.

(c) Prove or disprove: This estimate is an efficient estimator.

Problem 5. Let the independent normal random variables $Y_1, \ldots, Y_n$ have, respectively, the probability density functions $N(\mu, \gamma^2 x_i^2)$, $i = 1, \ldots, n$, where the given $x_1, \ldots, x_n$ are not all equal and no one of which is zero.

(a) Construct a confidence interval for γ with significance level $1 - \alpha$.

(b) Discuss the test of the hypothesis $H_0 : \gamma = 1, \mu$ unspecified, against all alternatives $H_1 : \gamma \neq 1, \mu$ unspecified.

S.-T. Yau College Student Mathematics Contests 2015

Geometry and Topology

Individual

Please solve 5 out of the following 6 problems.

1. Let n, m be positive integers. Show that the product of spheres $S^n \times S^m$ has trivial tangent bundle if and only if n or m is odd.

2. Show that there does not exist a compact three-dimensional manifold M whose boundary is the real projective space $\mathbb{RP}^2$.

3. Let M^n be a smooth manifold without boundary and X a smooth vector field on M. If X does not vanish at $p \in M$, show that there exists a local coordinate chart $(U; x_1, \ldots, x_n)$ centered at p such that in U the vector field X takes the form $X = \frac{\partial}{\partial x_1}$.

4. Let $M \to \mathbb{R}^3$ be a compact simply-connected closed surface. Prove that if M has constant mean curvature, then M is a standard sphere.

5. Let M be an n-dimensional compact Riemannian manifold with diameter π/c and Ricci curvature $\geq (n-1)c^2 > 0$. Show that M is isometric to the standard n-sphere in $\mathbb{R}^{n+1}$ with radius $1/c$.

6. Suppose (M, g) is a Riemannian manifold and $p \in M$. Show that the second-order Taylor series of g in normal coordinates centered at p is

$$g_{ij}(x) = \delta_{ij} - \frac{1}{3} \sum_{k,l} R_{iklj} x_k x_l + O(|x|^3).$$

S.-T. Yau College Student Mathematics Contests 2015

Algebra and Number Theory

Individual (5 problems)

This exam of 160 points is designed to test how much you know rather than how much you don't know. You are not expected to finish all problems but do as much as you can.

Problem 1. (20 pt) Let G be an finite $\mathbb{Z}$-module (i.e., a finite abelian group with additive group law) with a bilinear, (strongly) alternative, and non-degenerate pairing

$$\ell : G \times G \to \mathbb{Q}/\mathbb{Z}.$$

Here "(strongly) alternating" means for every $a \in G$, $\ell(a,a) = 0$; "non-degenerate" means for every nonzero $a \in G$ there is a $b \in G$ such that $\ell(a,b) \neq 0$. Show in steps the following statement:

(S) : *G is isomorphic to $H_1 \oplus H_2$ for some finite abelian groups $H_1 \simeq H_2$ such that $\ell|_{H_i \times H_i} = 0$.*

(1.1) (5pt) For every $a \in G$, write $o(a)$ for the order of a and $\ell_a : G \longrightarrow \mathbb{Q}/\mathbb{Z}$ for the homomorphism $\ell_a(b) = \ell(a,b)$. Show that the image of ℓ_a is $o(a)^{-1}\mathbb{Z}/\mathbb{Z}$.

(1.2) (5pt) Show that G has a pair of elements a, b with the following properties:

(a) $o(a)$ is maximal in the sense that for any $x \in G$, $o(x) \mid o(a)$;

(b) $\ell(a,b) = o(a)^{-1} \mod \mathbb{Z}$.

(c) $o(a) = o(b)$

We call the subgroup $< a, b >:= \mathbb{Z}a + \mathbb{Z}b$ a *maximal hyperbolic subgroup* of G.

(1.3) (5pt) Let $< a, b >$ be a maximal hyperbolic subgroup of G. Let G' be the orthogonal complement of $< a, b >$ consisting of elements $x \in G$ such that $\ell(x, c) = 0$ for all $c \in < a, b >$. Show that G is a direct sum as follows:

$$G = \mathbb{Z}a \oplus \mathbb{Z}b \oplus G'.$$

(1.4) (5pt) Finish the proof of (S) by induction.

Problem 2 (40pt)**.** Let $\mathrm{O}_n(\mathbb{C})$ denote the group of $n \times n$ orthogonal complex matrices, and $M_{n\times k}(\mathbb{C})$ the space of $n \times k$ complex matrices, where n and k are two positive integers. For $i = 0, 1$, let F_i be the space of rational function f on $\mathrm{M}_{n\times k}(\mathbb{C})$ such that

$$(*) \qquad f(gx) = \det(g)^i f(x) \quad \text{for all } g \in \mathrm{O}_n(\mathbb{C}) \text{ and } x \in M_{n\times k}(\mathbb{C}).$$

We want to study in steps the structures of F_0 and F_1.

(2.1) (10pt) For each $x \in M_{n\times k}$, let V_x denote the subspace of $\mathbb{C}^n$ generated by columns of x, and let $Q(x) = x^t \cdot x \in M_{k\times k}(\mathbb{C})$. Show the following are equivalent:

(a) the space V_x has dimension k, and the Euclidean inner product $(\cdot, \cdot)$ is non-degenerate on V_x in the sense that $V_x^\perp \cap V_x = 0$.

(b) $\det Q(x) \neq 0$.

(2.2) (10pt) Show that F_0 is a field generated by entries of $Q(x)$.

(2.3) (10pt) Assuem $k < n$ and let $f \in F_1$. Show that $f = 0$ by the following two steps:

(a) for any $x \in M_{n\times k}(\mathbb{C})$ with $\det Q(x) \neq 0$, construct a $g \in O_n(\mathbb{C})$ such that $g|_{V_x} = 1$ and $\det g = -1$.

(b) Show that f vanishes on a general point $x \in M_{n\times k}(\mathbb{C})$ with $\det Q(x) \neq 0$, thus $f \equiv 0$.

(2.4) (10pt) Assume $k \geq n$. Show that F_1 is a free vector space of rank 1 over F_0.

Problem 3. (40pt) Consider the equation $f(x) := x^3 + x + 1 = 0$. We want to show in steps that

for any prime p, if $\left(\frac{31}{p}\right) = -1$, then $x^3 + x + 1$ is solvable mod p.

Let x_1, x_2, x_3 be three roots of $f(x) := x^3 + x + 1 = 0$. Let $F = \mathbb{Q}(x_1)$, and $L = \mathbb{Q}(x_1, x_2, x_3)$, and $K = \mathbb{Q}(\sqrt{\Delta})$ where Δ is the discriminant of $f(x)$:

$$\Delta = [(x_1 - x_2)(x_2 - x_3)(x_3 - x_1)]^2.$$

(3.1) (10pt) Show that f is irreducible, that $\Delta = -31$, and that F is not Galois over $\mathbb{Q}$;

(3.2) (10pt) Show that $\mathrm{Gal}(L/\mathbb{Q}) \simeq S_3$, the permutation group of three letters, that $\mathrm{Gal}(L/K) \simeq \mathbb{Z}/3\mathbb{Z}$, and that $\mathrm{Gal}(L/F) \simeq \mathbb{Z}/2\mathbb{Z}$;

(3.3) (20pt) Let O_F, O_K, O_L = be rings of integers of F, K, L respectively. Let p be a prime such that $x^3 + x + 1 = 0$ is not soluble in $\mathbb{Z}/p\mathbb{Z}$. Show the following:

(a) (5pt) pO_F is still a prime ideal in O_F,

(b) (5pt) pO_L is product of two prime ideals in O_L, and

(c) (5pt) pO_K is product of two primes ideals in O_K, and

(d) (5pt) $x^2 + 31 = 0$ is soluble in $\mathbb{F}_p$.

Problem 4. (40pt) Let p be a prime and $\mathbb{Z}_p$ the ring of p-adic integers with a p-adic norm normalized by $|p| = p^{-1}$. Let $\phi : \mathbb{Z}_p \longrightarrow \mathbb{Z}_p$ be a map defined by a power series of the form

$$\phi(x) = x^p + p \sum a_n x^n, \qquad a_n \in \mathbb{Z}_p, \quad |a_n| \longrightarrow 0.$$

Let E be a field, and F the E-vector space of locally constant E-valued functions on $\mathbb{Z}_p$ with an operator ϕ^* defined by $\phi^* f = f \circ \phi$. We want to show in steps the following statement:

The set of eigenvalues of ϕ^* *on* F *is* $\{0, 1\}$.

(4.1) (10pt) Show that ϕ is a contraction map on each residue class $R \in \mathbb{Z}_p/p\mathbb{Z}_p$:

$$|\phi(x) - \phi(y)| \leq p^{-1}|x - y|, \qquad \forall x, y \in R.$$

(4.2) (10pt) Show that there is a $\epsilon_R \in R$ for each residue class R such that

$$\lim_n \phi^n(x) = \epsilon_R, \qquad \forall x \in R.$$

Here ϕ^n is defined inductively by $\phi^1 = \phi$, $\phi^n = \phi^{n-1} \circ \phi$.

(4.3) (10pt) Let F_0 (resp. F_1) be the subspace of functions f vanishing on each ϵ_R (resp. constant on R) for all residue class R. Show that $\phi^* = 1$ on F_1, and that for each $f \in F_0$ $\phi^{*n} f = 0$ for some $n \in \mathbb{N}$.

(4.4) (10pt) Show that for any $a \in E$, $a \neq 0, 1$, the operator $\phi^* - a$ is invertible on F.

Problem 5 (20pt)**.** Check if the following rings are UFD (unique factorization domain).

(5.1) (5pt) $R_1 = \mathbb{Z}[\sqrt{6}]$;

(5.2) (5pt) $R_2 = \mathbb{Z}[(1 + \sqrt{-11})/2]$;

(5.3) (5pt) $R_3 = \mathbb{C}[x, y]/(x^2 + y^2 - 1)$;

(5.4) (5pt) $R_4 = \mathbb{C}[x, y]/(x^3 + y^3 - 1)$.

S.-T. Yau College Student Mathematics Contests 2015

Applied Math. and Computational Math.

Individual (5 problems)

Problem 1. Let r and s be relatively prime positive integers. Prove that the number of lattice paths from $(0,0)$ to (r,s), which consists of steps $(1,0)$ and $(0,1)$ and never go above the line $ry = sx$ is given by

$$\frac{1}{r+s}\binom{r+s}{s}.$$

Problem 2. The following 2×2 block matrix

$$C(\alpha) = \begin{bmatrix} \alpha I & A \\ A^T & 0 \end{bmatrix}$$

plays a key role in an augmented system method to solve linear least squares problem, a fundamental numerical linear algebra problem for fitting a linear model to observations subject to errors in science, where $A \in \mathbf{R}^{m\times n}$ is of full rank $n \le m$, I is a $m\times m$ identity matrix, and $\alpha \ge 0$. Prove the following results which address the question of optimal choice of scaling α for stabiltiy of the augmented system method.

(a) The eigenvalues of $C(\alpha)$ are

$$\frac{\alpha}{2} \pm \left(\frac{\alpha^2}{4} + \sigma_i^2\right)^{1/2} \quad \text{for } i = 1, 2, \dots, n, \quad \text{and} \quad \alpha \quad (m-n \text{ times}),$$

where σ_i for $i = 1, 2, \dots, n$ are the singular values of A, arranged in the decreasing order, i.e., $\sigma_1 \ge \sigma_2 \ge \cdots \ge \sigma_n$.

(b) The condition number $\kappa_2(C(\alpha)) = \|C(\alpha)\|_2 \, \|[C(\alpha)]^{-1}\|_2$ has the following bounds:

$$\sqrt{2}\kappa_2(A) \le \min_\alpha \kappa_2(C(\alpha)) \le 2\kappa_2(A),$$

with $\min_\alpha \kappa_2(C(\alpha))$ being achieved for $\alpha = \sigma_n/\sqrt{2}$, and

$$\max_\alpha \kappa_2(C(\alpha)) > \kappa_2(A)^2,$$

where $\|\cdot\|$ is the spectral norm of a matrix.

Recall that any matrix $A \in \mathbf{R}^{m\times n}$ has a singular value decomposition (SVD):

$$A = U\Sigma V^T, \quad \Sigma = \operatorname{diag}(\sigma_1, \sigma_2, \dots, \sigma_p) \in \mathbf{R}^{m\times n}, \quad p = \min(m, n),$$

where $\sigma_1 \ge \sigma_2 \ge \cdots \ge \sigma_p \ge 0$, and $U \in \mathbf{R}^{m\times m}$, $V \in \mathbf{R}^{n\times n}$ are both orthogonal. The σ_i are the singular values of A and the columns of U and V are the left and right singular vectors of A, respectively.

Problem 3. Solve the following linear hyperbolic partial differential equation

$$u_t + au_x = 0, \quad t \geq 0, \tag{1}$$

where a is a constant. Using the finite difference approximation, we can obtain the forward-time central-space scheme as follows,

$$\frac{u_m^{n+1} - u_m^n}{k} + a\frac{u_{m+1}^n - u_{m-1}^n}{2h} = 0, \tag{2}$$

where k and h are temporal and spatial mesh sizes.

(a) Show that when we fix $\lambda = k/h$ as a positive constant, the forward-time central-space scheme (2) is consistent with equation (1).

(b) Analyze the stability of this method. Is the method stable with $\lambda = k/h$ being fixed as a constant?

(c) How would the answer change if you are allowed to make $\lambda = k/h$ small?

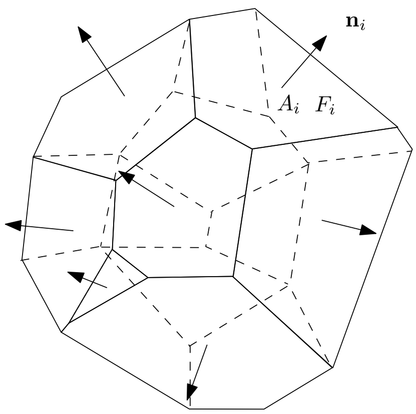
(d) Would this is a good scheme to use even if you can make it stable by making λ small? If not, please provide a simple modification to make this scheme stable by keeping λ fixed.

Problem 4. Let $A, H, Q \in \mathbb{C}^{n\times n}$ and Q is non-singular. Assume that $H = Q^{-1}AQ$ and H is properly upper Hessenberg. Show that

$$\text{span}\{q_1, q_2, \ldots, q_j\} = \mathcal{K}_j(A, q_1), \qquad j = 1, 2, \ldots, n$$

where q_j is the j-th column of Q, and $\mathcal{K}_j(A, q_1) = \text{span}\{q_1, Aq_1, \ldots, A^{j-1}q_1\}$.

Problem 5. Minkowski Problem.



Assume P is a convex polyhedron embedded in $\mathbb{R}^3$, the faces are $\{F_1, F_2, \cdots, F_k\}$, the unit normal vector to the face F_i is $\mathbf{n}_i$, the area of F_i is A_i, $1 \leq i \leq k$.

- Show that

$$A_1\mathbf{n}_1 + A_2\mathbf{n}_2 + \cdots A_k\mathbf{n}_k = \mathbf{0}, \tag{3}$$

- Given k unit vectors $\{\mathbf{n}_1, \mathbf{n}_2, \cdots, \mathbf{n}_k\}$ which can not be contained in any half space, and k real positive numbers $\{A_1, A_2, \cdots, A_k\}$, $A_i > 0$, and satisfying the condition (3), show that there exists a convex polyhedron P, whose face normals are $\mathbf{n}_i$'s, face areas are A_i's.

S.-T. Yau College Student Mathematics Contests 2015

Analysis and Differential Equations

Team

Please solve 5 out of the following 6 problems.

1. Let $\phi \in C([a,b],R)$. Suppose for every function $h \in C^1([a,b],R), h(a) = h(b) = 0$, we have

$$\int_a^b \phi(x)h(x)dx = 0.$$

Prove that $\phi(x) = 0$.

2. Let f be a Lebesgue integrable function over $[a, b+\delta], \delta > 0$, prove that

$$\lim_{h\to 0+} \int_a^b |f(x+h) - f(x)|dx \to 0.$$

3. Let $L(q, q^{'}, t)$ be a function of $(q, q^{'}, t) \in TU \times R, U$ is an open domain in R^n. Let $\gamma : [a,b] \to U$ be a curve in U. Define a functional $S(\gamma) = \int_a^b L(\gamma(t), \gamma^{'}(t), t)dt$. We say that γ is an extremal if for every smooth variation of $\gamma, \phi(t,s), s \in (-\delta, \delta), \phi(t, 0) = \gamma(t), \phi_s = \phi(t,s)$, we have $\frac{dS(\phi_s)}{ds}|_{s=0} = 0$. Prove that every extremal γ satisfies the Euler-Lagrange equation: $\frac{d}{dt}(\frac{\partial L}{\partial q^{'}}) = \frac{\partial L}{\partial q}$.

4. Let $f : U \to U$ be a holomorphic function with U a bounded domain in the complex plane. Assuming $0 \in U, f(0) = 0, f^{'}(0) = 1$, prove that $f(z) = z$.

5. Let $T : H_1 \to H_2$ be a bounded operator of Hilbert spaces H_1, H_2. Let $S : H_1 \to H_2$ be a compact operator, that is, for every bounded sequence $\{v_n\} \in H_1, Sv_n$ has a converging subsequence. Show that $Coker(T+S) = H_2/\overline{Im(T+S)}$ is finite dimensional and $Im(T+S)$ is closed in H_2. (Hint: Consider equivalent statements in terms of adjoint operators.)

6. Let $u \in C^2(\bar{\Omega}), \Omega \subset R^d$ is a bounded domain with a smooth boundary.

1) Let u be a solution of the equation $\Delta u = f, u|_{\partial\Omega} = 0, f \in L^2(\Omega)$. Prove that there is a constant C depends only Ω such that

$$\int_\Omega (\Sigma_{j=1}^n (\frac{\partial u}{\partial x_j})^2 + u^2)dx \le C \int_\Omega f^2(x)dx.$$

2) Let $\{u_n\}$ be a sequence of harmonic functions on Ω, such that $||u_n||_{L^2(\Omega)} \le M < \infty$, for a constant M independent of n. Prove that there is a converging subsequence $\{u_{n_k}\}$ in $L^2(\Omega)$.

S.-T. Yau College Student Mathematics Contests 2015

Probability and Statistics

Team (5 problems)

Problem 1. One hundred passengers board a plane with exactly 100 seats. The first passenger takes a seat at random. The second passenger takes his own seat if it is available, otherwise he takes at random a seat among the available ones. The third passenger takes his own seat if it is available, otherwise he takes at random a seat among the available ones. This process continues until all the 100 passengers have boarded the plane. What is the probability that the last passenger takes his own seat?

Problem 2. Assume a sequence of random variables X_n converges in distribution to a random variable X. Let $\{N_t, t \geq 0\}$ be a set of positive integer-valued random variables, which is independent of (X_n) and converges in probability to ∞ as $t \to \infty$. Prove that X_{N_t} converges in distribution to X as $t \to \infty$.

Problem 3. Suppose $T_1, T_2, \ldots, T_n$ is a sequence of independent, identically distributed random variables with the exponential distribution of the density function

$$p(x) = \begin{cases} e^{-x}, & x \geq 0; \\ 0, & x < 0. \end{cases}$$

Let $S_n = T_1 + T_2 + \cdots + T_n$. Find the distribution of the random vector

$$V_n = \left\{ \frac{T_1}{S_n}, \frac{T_2}{S_n}, \cdots, \frac{T_n}{S_n} \right\}.$$

Problem 4. Suppose that X and Z are jointly normal with mean zero and standard deviation 1. For a strictly monotonic function $f(\cdot)$, $\text{cov}(X, Z) = 0$ if and only if $\text{cov}(X, f(Z)) = 0$, provided the latter covariance exists. **Hint:** Z can be expressed as $Z = \rho X + \varepsilon$ where X and ε are independent and $\varepsilon \sim N(0, \sqrt{1-\rho^2})$.

Problem 5. Consider the following penalized least-squares problem (Lasso):

$$\frac{1}{2}\|\mathbf{Y} - \mathbf{X}\beta\|^2 + \lambda\|\beta\|_1$$

Let $\widehat{\beta}$ be a minimizer and $\boldsymbol{\Delta} = \widehat{\beta} - \beta^*$ for any given β^*. If $\lambda > 2\|\mathbf{X}^T(\mathbf{Y} - \mathbf{X}\beta^*)\|_\infty$, show that

1. $\|\mathbf{Y} - \mathbf{X}^T\widehat{\beta}\|^2 - \|\mathbf{Y} - \mathbf{X}^T\beta^*\|^2 > -\lambda\|\boldsymbol{\Delta}\|_1$.

2. $\|\mathbf{\Delta}_{S^c}\|_1 \leq 3\|\mathbf{\Delta}_S\|_1$, where $S = \{j : \beta_j^* \neq 0\}$ is the support of the vector β^*, S^c is its complement set, $\mathbf{\Delta}_S$ is the subvector of $\mathbf{\Delta}$ restricted on the set S, and $\|\mathbf{\Delta}_S\|_1$ is its L_1-norm.

S.-T. Yau College Student Mathematics Contests 2015

Geometry and Topology

Team

Please solve 5 out of the following 6 problems.

1. Let $SO(3)$ be the set of all 3×3 real matrices A with determinant 1 and satisfying ${}^tAA = I$, where I is the identity matrix and ${}^tA$ is the transpose of A. Show that $SO(3)$ is a smooth manifold, and find its fundamental group. You need to prove your claims.

2. Let X be a topological space. The *suspension* $S(X)$ of X is the space obtained from $X \times [0,1]$ by contracting $X \times \{0\}$ to a point and contracting $X \times \{1\}$ to another point. Describe the relation between the homology groups of X and $S(X)$.

3. Let $F : M \to N$ be a smooth map between two manifolds. Let X_1, X_2 be smooth vector fields on M and let Y_1, Y_2 be smooth vector fields on N. Prove that if $Y_1 = F_*X_1$ and $Y_2 = F_*X_2$, then $F_*[X_1, X_2] = [Y_1, Y_2]$, where $[\ ,\ ]$ is the Lie bracket.

4. Let M_1 and M_2 be two compact convex closed surfaces in $\mathbb{R}^3$, and $f : M_1 \to M_2$ a diffeomerphism such that M_1 and M_2 have the same inner normal vectors and Gauss curvatures at the corresponding points. Prove that f is a translation.

5. Prove the second Bianchi identity:

$$R_{ijkl,h} + R_{ijlh,k} + R_{ijhk,l} = 0$$

6. Let M_1, M_2 be two complete n-dimensional Riemannian manifolds and $\gamma_i : [0,a] \to M_i$ are two arc length parametrized geodesics. Let ρ_i be the distance function to $\gamma_i(0)$ on M_i. Assume that $\gamma_i(a)$ is within the cut locus of $\gamma_i(0)$ and for any $0 \le t \le a$ we have the inequality of sectional curvatures

$$K_1(X_1, \frac{\partial}{\partial \gamma_1}) \ge K_2(X_2, \frac{\partial}{\partial \gamma_2}),$$

where $X_i \in T_{\gamma_i(t)}M_i$ is any unit vector orthogonal to the tangent $\frac{\partial}{\partial \gamma_i}$.

Then

$$Hess(\rho_1)(\widetilde{X}_1, \widetilde{X}_1) \le Hess(\rho_2)(\widetilde{X}_2, \widetilde{X}_2),$$

where $\widetilde{X}_i \in T_{\gamma_i(a)}M_i$ is any unit vector orthogonal to the tangent $\frac{\partial}{\partial \gamma_i}(a)$.

S.-T. Yau College Student Mathematics Contests 2015

Algebra and Number Theory

Team

This exam contains 6 problems. Please choose 5 of them to work on.

Problem 1. (20pt) Let $V = \mathbb{R}^n$ be an Euclidean space equipped with usual inner product, and g an orthogonal matrix acting on V. For $a \in V$, let s_a denote the reflection

$$s_a(x) := x - 2\frac{(x,a)}{(a,a)}a, \qquad \forall x \in V.$$

(1.1) (10pt) For $a = (g-1)b \neq 0$, show that

$$\ker(s_a g - 1) = \ker(g-1) \oplus \mathbb{R}b.$$

(1.2) (10pt) Show that g is a product of $\dim[(g-1)V]$ reflections.

Problem 2. (20pt) Let p and q be two distinct prime numbers. Let G be a non-abelian finite group satisfying the following conditions:

1. all nontrivial elements have order either p or q;
2. The q-Sylow subgroup H_q is normal and is a nontrivial abelian group.

Show in steps the following statement:

The group G is of the form $(\mathbb{Z}/p\mathbb{Z}) \ltimes (\mathbb{Z}/q\mathbb{Z})^n$, where the action of $1 \in \mathbb{Z}/p\mathbb{Z}$ on $(\mathbb{Z}/q\mathbb{Z})^n \simeq \mathbb{F}_q^n$ is given by a matrix $M(1) \in \mathrm{GL}_n(\mathbb{F}_q)$ whose eigenvalues are all primitive *p-th roots of unities.*

(2.1) (5pt) Let H_p denote a p-Sylow subgroup of G. Show that its inclusion into G induces an isomorphism $H_p \cong G/H_q$, and that $G \simeq H_p \ltimes H_q$.

(2.2) (5pt) Let $M : H_p \longrightarrow \mathrm{Aut}(H_q) \simeq \mathrm{GL}_n(\mathbb{F}_q)$ be the homomorphism induced from the conjugations. Show that for each $1 \neq a \in H_p$, $M(a)$ is semisimple whose eigenvalues are all *primitive* p-th roots of unities. In particular M is injective.

(2.3) (5pt) Show that if two nontrivial elements $a, b \in H_p$ commute with each other, then $a = b^n$ for some $n \in \mathbb{N}$, and that $H_p \simeq \mathbb{Z}/p\mathbb{Z}$.

(2.4) (5pt) Complete the solution of the problem.

Problem 3. (20pt) Let ζ be a root of unity satisfying an equation $\zeta = 1 + N\eta$ for an integer $N \geq 3$ and an algebraic integer η. Show that $\zeta = 1$.

Problem 4. (20pt) Let G be a finite group and (π, V) a finite dimensional $\mathbb{C}G$-module. For $n \geq 0$, let $\mathbb{C}[V]_n$ be the space of homogeneous polynomial functions on V of degree n. For a simple G-representation ρ, denote by $a_n(\rho)$ the multiplicity of ρ in $\mathbb{C}[V]_n$. Show that

$$\sum_{n\geq 0} a_n(\rho)t^n = \frac{1}{|G|}\sum_{g\in G}\frac{\overline{\chi_\rho(g)}}{\det(\mathrm{id}_V - \pi(g)t)}.$$

Problem 5. (20pt) Let A be an $n \times n$ complex matrix considered as an operator on $V = (\mathbb{C}^n, (\cdot,\cdot))$ with standard hermitian form. Let $A^* = \bar{A}^t$ be the hermitian transpose of A:

$$(Ax, y) = (x, A^*y), \qquad \forall x, y \in \mathbb{C}^n.$$

(5.1) (5pt) For any $\lambda \in \mathbb{C}$, show the identity:

$$\ker(A - \lambda)^\perp = (A^* - \bar{\lambda})V.$$

(5.2) (15pt) Show the equivalence of the following two statements:

(a) A commutes with A^*;

(b) there is a unitary matrix U (in the sense $U^* = U^{-1}$), such that UAU^{-1} is diagonal.

Problem 6. (20pt) Consider the polynomial $f(x) = x^5 - 80x + 5$.

(6.1) (5pt) Show that f is irreducible over $\mathbb{Q}$;

(6.2) (15 pt) Show in steps that the split field K of f has Galois group $G := \mathrm{Gal}(K/\mathbb{Q})$ isomorphic to S_5, the symmetric group of 5 letters.

(a) (5pt) $f = 0$ has exactly two complex roots;

(b) (5pt) G can be embedded into S_5 with image containing cycles (12345) and (12);

(c) (5pt) $G \simeq S_5$.

S.-T. Yau College Student Mathematics Contests 2015

Applied Math. and Computational Math.

Team (5 problems)

Problem 1. Consider the elliptic interface problem

$$(a(x)u_x)_x = f, \ x \in (0,1)$$

with the Dirichlet boundary condition

$$u(0) = u(1) = 0.$$

Here, f is a smooth function, the elliptic coefficient $a(x)$ is discontinuous across an interface point ξ, that is,

$$a(x) = \begin{cases} a_0 & \text{for } 0 < x < \xi \\ a_1 & \text{for } \xi < x < 1, \end{cases}$$

$a_0, a_1 > 0$ are positive constants, and $0 < \xi < 1$ is an interface point. Across the interface, we need to impose two jump conditions

$$u(\xi-) = u(\xi+), \ a(\xi-)u_x(\xi-) = a(\xi+)u_x(\xi+).$$

Question:

1. (25%) Design a numerical method to solve this problem. The method should be at least first order. It is better to be high order (if your method is first order, you get 20% points).
2. (75%) Prove your accuracy and convergence arguments (if your method is first order, you get 60% points).

Problem 2. Let G be graph of a social network, where for each pair of members there is either no connection, or a positive or a negative one.

An unbalanced cycle in G is a a cycle which have odd number of negative edges. Traversing along such a cycle with social rules such as friend of enemy are enemy would result in having a negative relation of one with himself!

A resigning in G at a vertex v of G is to switch the type (positive or negative) of all edges incident to v.

Question: Show that one can switch all edge of G into positive edges using a sequence resigning if and only if there is no unbalanced cycle in G.

Problem 3. We consider particles which are able to produce new particles of like kind. A single particle forms the original, or zero, generation. Every particle has probability p_k $(k = 0, 1, 2, \dots)$ of creating exactly k new particles; the direct descendants of the nth generation form the $(n+1)$st generation. The particles of each generation act independently of each other.

Assume $0 < p_0 < 1$. Let $P(x) = \sum_{k\geq 0} p_k x^k$ and $\mu = P'(1) = \sum_{k\geq 0} kp_k$ be the expected number of direct descendants of one particle. Prove that if $\mu > 1$, then the probability x_n that the process terminates at or before the nth generation tends to the unique root $\sigma \in (0,1)$ of equation $\sigma = P(\sigma)$.

Problem 4. (Isopermetric inequality). Consider a closed plane curve described by a parametric equation $(x(t), y(t)), 0 \leq t \leq T$ with parameter t oriented counterclockwise and $(x(0), y(0)) = (x(T), y(T))$.

(a): Show that the total length of the curve is given by

$$L = \int_0^T \sqrt{(x'(t))^2 + (y'(t))^2)}\, dt$$

(b): Show that the total area enclosed by the curve is given by

$$A = \frac{1}{2} \int_0^T \big(x(t)y'(t) - y(t)x'(t)\big)\, dt$$

(c): The classical iso-perimetric inequality states that for closed plane curves with a fixed length L, circles have the largest enclosed area A. Formulate this question into a variational problem.

(d): Derive the Euler-Lagrange equation for the variational problem in **(c)**.

(e): Show that there are two constants x_0 and y_0 such that

$$(x(t) - x_0)^2 + (y(t) - y_0)^2 \equiv r^2$$

where $r = L/(2\pi)$. Explain your result.

Problem 5. Let $A \in \mathbb{R}^{n\times m}$ with rank $r < \min(m, n)$. Let $A = U\Sigma V^T$ be the SVD of A, with singular values $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0$.

(a) Show that, for every $\epsilon > 0$, there is a full rank matrix $A_\epsilon \in \mathbb{R}^{n\times m}$ such that $||A - A_\epsilon||_2 = \epsilon$.

(b) Let $A_k = U\Sigma_k V^T$ where $\Sigma_k = \text{diag}(\sigma_1, \ldots, \sigma_k, 0, \ldots, 0)$ and $1 \leq k \leq r-1$. Show that $\text{rank}(A_k) = k$ and

$$\sigma_{k+1} = ||A - A_k||_2 = \min\{||A - B||_2 \quad | \quad \text{rank}(B) \leq k\}$$

(c) Assume that $r = \min(m, n)$. Let $B \in \mathbb{R}^{n\times m}$ and assume that $||A - B||_2 < \sigma_r$. Show that $\text{rank}(B) = r$.

S.-T. Yau College Student Mathematics Contests 2016

Analysis and Differential Equations

Individual

Please solve 5 out of the following 6 problems.

1. Suppose that F is continuous on $[a, b], F^{'}(x)$ exists for every $x \in (a, b), F^{'}(x)$ is integrable. Prove that F is absolutely continuous and

$$F(b) - F(a) = \int_a^b F^{'}(x)dx.$$

2. Suppose that f is integrable on $\mathbf{R}^n$, let $K_\delta(x) = \delta^{-\frac{n}{2}} e^{\frac{-\pi|x|^2}{\delta}}$ for each $\delta > 0$. Prove that the convolution

$$(f * K_\delta)(x) = \int_{\mathbf{R}^n} f(x-y)K_\delta(y)dy$$

is integrable and $||(f * K_\delta) - f||_{L^1(\mathbf{R}^n)} \to 0$, as $\delta \to 0$.

3. Prove that a bounded function on interval $I = [a, b]$ is Riemann integrable if and only if its set of discontinuities has measure zero. You may prove this by the following steps.

Define $I(c, r) = (c - r, c + r), osc(f, c, r) = \sup_{x,y \in J \cap I(c,r)} |f(x) - f(y)|, osc(f, c) = \lim_{r \to 0} osc(f, r, c)$.

1) f is continuous at $c \in J$ if and only if $osc(f, c) = 0$.

2) For arbitrary $\epsilon > 0, \{c \in J | osc(f, c) \geq \epsilon\}$ is compact.

3) If the set of discontinuities of f has measure 0, then f is Riemann integrable.

4. 1) Let f be the Rukowski map: $w = \frac{1}{2}(z + \frac{1}{z})$. Show that it maps $\{z \in \bar{\mathbf{C}} || z| > 1\}$ to $\bar{\mathbf{C}}/[-1, 1], \bar{\mathbf{C}} = \mathbf{C} \cup \{\infty\}$.

2) Compute the integral:

$$\int_0^\infty \frac{\log x}{x^2 - 1} dx.$$

5. Let f be a doubly periodic meromorphic function over the complex plane, i.e. $f(z + 1) = f(z), f(z + i) = f(z), z \in \mathbf{C}$, prove that the number of zeros and the number of poles are equal.

6. Let A be a bounded self-adjoint operator over a complex Hilbert space. Prove that the spectrum of A is a bounded closed subset of the real line $\mathbf{R}$.

S.-T. Yau College Student Mathematics Contests 2016

Probability and Statistics

Individual (5 problems)

Problem 1. A random walker moves on the lattice $\mathbb{Z}^2$ according to the following rule: in the first step it moves to one of its neighbors with probability 1/4, and then in step $n > 1$ it moves to one of the neighbors that it didn't visit in the step $n - 1$ with equal probability. Let T be the time when the random walker steps on a site that it already visited. Please show that the expectation of T is less than 35.

Problem 2. Let X be a $N \times N$ random matrix with i.i.d. random entries, and

$$\mathbb{P}(X_{11} = 1) = \mathbb{P}(X_{11} = -1) = 1/2$$

Define

$$\|X\|_{op} = \sup_{\mathbf{v} \in \mathbb{C}^N : \|\mathbf{v}\|_2 = 1} \|X\mathbf{v}\|_2$$

Please show that for any fixed $\delta > 0$,

$$\lim_{N \to \infty} \mathbb{P}(\|X\|_{op} \geq N^{1/2+\delta}) = 0$$

Hint: $\|X\|_{op}^2 \leq \mathrm{tr}|X|^2$

Problem 3. Suppose that 2016 balls are put into 2016 boxes with each ball independently being put into box i with probability $\frac{1}{3 \times 1008}$ for $i \leq 1008$ and $\frac{2}{3 \times 1008}$ for $i > 1008$. Let T be the number of boxes containing exactly 2 balls. Please find the variance of T.

Problem 4. Let $b > a > 0$ be real numbers. Let X be a random variable taking values in $[a, b]$, and let $Y = \frac{1}{X}$. Determine the set of all possible values of $\mathbb{E}(X) \times \mathbb{E}(Y)$.

Problem 5. Let X_1, X_2, ... be independent and identically distributed real-valued random variables such that $\mathbb{E}(X_1) = -1$. Let $S_n = X_1 + \cdots + X_n$ for all $n \geq 1$, and let T be the total number of $n \geq 1$ satisfying $S_n \geq 0$. Compute $P(T = \infty)$.

S.-T. Yau College Student Mathematics Contests 2016

Geometry and Topology

Individual

Please solve 5 out of the following 6 problems.

1. Let M be a compact odd-dimensional manifold with boundary ∂M. Show that the Euler characteristics of M and ∂M are related by:

$$\chi(M) = \frac{1}{2}\chi(\partial M).$$

2. Compute the de Rham cohomology of a punctured two-dimensional torus $T^2 - \{p\}$, where $p \in T^2$. If $T^2 = \mathbb{R}^2/\mathbb{Z}^2$ with coordinates $(x, y) \in \mathbb{R}^2$, then is the volume form $\omega = dx \wedge dy$ exact?

3. Let $M^n \to \mathbb{R}^{n+1}$ be a closed oriented hypersurface. The r-th mean curvature of M^n is defined by

$$H_r := \frac{1}{\binom{n}{r}} \sum_{i_1 < i_2 \cdots < i_r} \lambda_{i_1}\lambda_{i_2}\cdots\lambda_{i_r}, \quad (1 \leq r \leq n)$$

where λ_i $(i = 1, \cdots, n)$ are principal curvatures of M^n. Prove that if all of λ_i are positive and H_r =constant for a certain r, then M^n is a hypersphere in $\mathbb{R}^{n+1}$.

4. State and prove the cut-off function lemma on a differentiable manifold.

5. Let M be a compact Riemannian manifold without boundary. Show that if M has positive Ricci curvature, then $H^1(M, \mathbb{R}) = 0$.

6. Let M be an orientable, closed and embedded minimal hypersurface in S^{n+1}. Denote by λ_1 the first eigenvalue for the Laplace-Beltrami operator on M. Prove that $\lambda_1 \geq n/2$.

Algebra and Number Theory

Individual

This test has 5 problems and is worth 100 points. Carefully justify your answers.

Problem 1 (20 points). Let E be a linear space over $\mathbb{R}$, of finite dimension $n \geq 2$, equipped with a positive definite symmetric bilinear form $\langle \cdot, \cdot \rangle$. Let $u_1, u_2, \ldots, u_n$ be a basis of E. Let $v_1, v_2, \ldots, v_n$ be the dual basis, that is,

$$\langle u_i, v_j \rangle = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j, \end{cases}$$

for all $i, j = 1, 2, \ldots, n$.

(a) (8 points) Assume that $\langle u_i, u_j \rangle \leq 0$ for all $1 \leq i < j \leq n$. Show that there is an orthogonal basis $u'_1, u'_2, \ldots, u'_n$ of E such that u'_i is a non-negative linear combination of $u_1, u_2, \ldots, u_i$, for all $i = 1, 2, \ldots, n$.

(b) (6 points) With the same assumption as in Part (a), show that $\langle v_i, v_j \rangle \geq 0$ for all $1 \leq i < j \leq n$.

(c) (6 points) Assume that $n \geq 3$. Show that the condition $\langle u_i, u_j \rangle \geq 0$ for all $1 \leq i < j \leq n$ does not imply that $\langle v_i, v_j \rangle \leq 0$ for all $1 \leq i < j \leq n$.

Problem 2 (20 points). Let $d \geq 1$ and $n \geq 1$ be integers.

(a) (5 points) Show that there are only finitely many subgroups $G \subseteq \mathbb{Z}^d$ of index n. Let $f_d(n)$ denote the number of such subgroups.

(b) (5 points) Let $g_d(n)$ denote the number of subgroups $H \subseteq \mathbb{Z}^d$ of index n such that the quotient group is cyclic. Show that $f_d(mn) = f_d(m)f_d(n)$ and $g_d(mn) = g_d(m)g_d(n)$ for coprime positive integers m and n.

(c) (5 points) Compute $g_d(p^r)$ for every prime power p^r, $r \geq 1$.

(d) (5 points) Compute $f_2(20)$.

Problem 3 (20 points). Let A be a complex $m \times m$ matrix. Assume that there exists an integer $N \geq 0$ such that $t_n = \operatorname{tr}(A^n)$ is an algebraic integer for all $n \geq N$. The goal of this problem is to show that the eigenvalues $a_1, \ldots, a_m$ of A are algebraic integers.

(a) (10 points) Show that there exist algebraic numbers $b_{ij} \in \mathbb{C}$, $1 \leq i, j \leq m$ such that

$$a_i^n = \sum_{j=1}^{m} b_{ij} t_{n+j-1}$$

for all $n \geq 0$ and all $1 \leq i \leq m$. In particular, $a_1, \ldots, a_m$ are algebraic numbers.

(b) (8 points) Let R be the ring of all algebraic integers in $\mathbb{C}$ and let K be the field of all algebraic numbers in $\mathbb{C}$. Show that for $a \in K$, if $R[a]$ is contained in a finitely-generated R-submodule of K, then $a \in R$.

(c) (2 points) Conclude that $a_1, \ldots, a_m$ are algebraic integers.

Problem 4 (20 points)**.** Let E be a Euclidean plane. For each line l in E, write $s_l \in \mathrm{Iso}(E)$ for the reflection with respect to l, where $\mathrm{Iso}(E)$ denotes the group of distance-preserving bijections from E to itself.

(a) (6 points) Let l_1 and l_2 be two distinct lines in E. Find the necessary and sufficient condition that s_{l_1} and s_{l_2} generate a finite group.

(b) (7 points) Let l_1, l_2 and l_3 be three pairwise distinct lines in E. Assume that s_{l_1}, s_{l_2} and s_{l_3} generate a finite group. Show that l_1, l_2, l_3 intersect at a point.

(c) (7 points) Let G be a finite subgroup of $\mathrm{Iso}(E)$ generated by reflections. Show that G is generated by at most two reflections.

Problem 5 (20 points)**.** Let G be a finite group of order $2^n m$ where $n \geq 1$ and m is an odd integer. Assume that G has an element of order 2^n. The goal of this problem is to show that G has a normal subgroup of order m.

(a) (5 points) Show that if M is a normal subgroup of G of order m, then M is the only subgroup of G of order m.

(b) (5 points) Let N be a normal subgroup of G and let P be a 2-Sylow subgroup of G. Show that $P \cap N$ is a 2-Sylow subgroup of N.

(c) (5 points) Show that the homomorphism $G \to \{\pm 1\}$ carrying g to $\mathrm{sgn}(l_g)$ is surjective. Here $\mathrm{sgn}(l_g)$ denotes the sign of the permutation $l_g \colon G \to G$ given by left multiplication by g.

(d) (5 points) Deduce by induction on n that G has a normal subgroup of order m.

S.-T. Yau College Student Mathematics Contests 2016

Applied Math. and Computational Math.

Individual (5 problems)

Problem 1. Consider the implicit leapfrog scheme

$$\frac{u_m^{n+1} - u_m^{n-1}}{2k} + a\left(1 + \frac{h^2}{6}\delta^2\right)^{-1} \delta_0 u_m^n = f_m^n$$

for the one-way wave equation

$$u_t + a u_x = f.$$

Here δ^2 is the central second difference operator, and δ_0 is the central first difference operator.

(1) show that the scheme is of order $(2, 4)$.
(2) show that the scheme is stable if and only if $|\frac{ak}{h}| < \frac{1}{\sqrt{3}}$.

Problem 2. A simple version of an enzyme-mediate chemical reaction process is given by

$$\mathrm{S} + \mathrm{E} \underset{k_2}{\overset{k_1}{\longleftrightarrow}} \mathrm{C} \xrightarrow{k_3} \mathrm{P} + \mathrm{E}$$

where S is the substrate reactant and P is the concentration of the desired product. An enzyme (or catalyst) E is a compound whose special property is that it allows for intermediate reaction steps that lead to a the overall reaction,

$$\mathrm{S} \longrightarrow \mathrm{P}.$$

Assume the initial conditions

$$S(0) = S_0, \quad E(0) = E_0, \quad C(0) = 0, \quad P(0) = 0;$$

k_1, k_2, k_3 are reaction rate constants.

(a) Convert the chemical reaction equation into a system of rate equations (ODEs) for $S(T)$, $E(T)$, $C(T)$, and $P(T)$ where T is the dimensional time. Nondimensionalize the equations using the scalings

$$T = t/(k_1 E_0), \quad S(T) = S_0 s(t), \quad P(T) = S_0 p(t), \quad E(T) = E_0 s(t), \quad C(T) = E_0 c(t),$$

$$\epsilon = \frac{E_0}{S_0} \ll 1, \quad \lambda = \frac{k_2}{k_1 S_0}, \quad \mu = \frac{k_2 + k_3}{k_1 S_0}.$$

(b) Use the expansions $s(t) = s_0(t) + \epsilon s_1(t) + O(\epsilon^2)$, $c(t) = c_0(t) + \epsilon c_1(t) + O(\epsilon^2)$, etc to determine the equations for the leading order slow solution. Show that $s_0(t)$ and $p_0(t)$ satisfies the following Michaelis-Menten equations

$$\dot{s}_0(t) = -(\mu - \lambda)\frac{s_0}{\mu + s_0}, \quad \dot{p}_0(t) = (\mu - \lambda)\frac{s_0}{\mu + s_0}.$$

Problem 3. We say that a vector $\mathbf{u} = (u_1, \ldots, u_n) \in \mathbb{N}^n$ is *multiplicatively dependent* if there is a non-zero vector $\mathbf{k} = (k_1, \ldots, k_n) \in \mathbb{Z}^n$ for which

$$u_1^{k_1} \cdots u_n^{k_n} = 1. \tag{1}$$

This notion plays a very important role in many number theoretic algorithms, such as *factorisation* and *primality testing*. It also (in a more general form) appears in some questions in *algebraic dynamics*. However the algorithm to decide whether $\mathbf{u}$ is multiplicatively dependent is not immediately obvious. The following statement *informally* means that if $\mathbf{u}$ is multiplicatively dependent the exponents $k_1, \ldots, k_n$ can be chosen to be reasonably small. Prove that if $\mathbf{u} = (u_1, \ldots, u_n) \in \mathbb{N}^n$ is multiplicatively dependent with $\|\mathbf{u}\|_\infty \le H$ where $\|\mathbf{u}\|_\infty = \max_{1\le i\le n} |u_i|$, then there is a non-zero vector $\mathbf{k} = (k_1, \ldots, k_n) \in \mathbb{Z}^n$ with

$$\|\mathbf{k}\|_\infty \le \left(\frac{2n \log H}{\log 2}\right)^{n-1}$$

(and hence for a fixed n it can be found in polynomial time of order $(\log H)^{n(n-1)}$).
Comment: To solve this problem, you can use the following statement (without proof) which *informally* means that if a system of homogeneous equations with integer coefficients has a nontrivial solution then it has an integer solutions with reasonably small components. It is required in many applications.
Let $A = (a_{ij})_{i,j=1}^{m,n}$ be an $m \times n$ matrix of rank $r \le n-1$ with integer entries of size at most H, that is,

$$|a_{ij}| \le H, \quad 1 \le i \le m, \ 1 \le j \le n.$$

Then there is an integer **non-zero** vector $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{Z}^n$ such that $A\mathbf{x} = \mathbf{0}$ and

$$\|\mathbf{x}\|_\infty \le (2nH)^{n-1}$$

where $\|\mathbf{x}\|_\infty = \max_{1\le i\le n} |x_i|$.

Problem 4. Consider a symmetric matrix $A_{n\times n}$, and let λ_i be a simple eigenvalue of A with

$$|\lambda_j - \lambda_i| = O(1), \quad j \ne i.$$

In inverse iteration of compute eigenvalue and eigenvector, one needs to solve the following linear system

$$(A - \mu I) y_{k+1} = x_k,$$

where μ is an approximation of eigenvalue λ_i, $\|x_k\| = 1$and obtain

$$x_{k+1} = \frac{y_{k+1}}{\|y_{k+1}\|}.$$

However, for μ close to λ_i, $A - \mu I$ has a very small eigenvalue and the linear system will be ill-conditioned. So there may be large error in the numerical solution to the linear system, denoted by $\tilde{y}_{k+1}$. Even though we may get large error in $\tilde{y}_{k+1}$, the $\tilde{x}_{k+1}$ we get from $\tilde{x}_{k+1} = \frac{\tilde{y}_{k+1}}{\|\tilde{y}_{k+1}\|}$ is accurate.

(1) $\tilde{y}_{k+1}$ satisfies

$$(A - \mu I + \delta A)\tilde{y}_{k+1} = x_k,$$

where $\|\delta A\| = O(\epsilon)$ and ϵ is the machine precision. Show that

$$(A - \lambda_i)\frac{\tilde{y}_{k+1}}{\|\tilde{y}_{k+1}\|}\| \le |\mu - \lambda_i| + \|\delta A\| + \frac{1}{\|\tilde{y}_{k+1}\|}.$$

(2) Let $\alpha_i = x_k^t q_i$, where q_i is the normalized eigenvector corresponding to λ_i. Show that

$$\|\tilde{y}_{k+1}\| \ge \frac{|\alpha_i|}{|\mu - \lambda_i| + \|\delta A\|}.$$

(3) Conclude that

$$\|x_{k+1} - (\pm)q_i\| = O(|\lambda_i - \mu| + \epsilon).$$

Problem 5. A function $f : \mathbb{R}^n \to \mathbb{R}$ in C^2 is called strongly convex if its Hessian matrix satisfies $\nabla^2 f \succeq mI$ for some $m > 0$. Show that the following statements are equivalent:

(a) f is strongly convex, i.e. $\nabla^2 f(x) \succeq mI$ for all $x \in \mathbb{R}^n$;

(b) For any $t \in [0, 1]$, any $x, y \in \mathbb{R}$,

$$f(tx + (1-t)y) \le tf(x) + (1-t)f(y) - \frac{m}{2}t(1-t)\|x - y\|^2;$$

(c) $\langle \nabla f(x) - \nabla f(y), x - y\rangle \ge m\|x - y\|^2$ for any $x, y \in \mathbb{R}^n$.

S.-T. Yau College Student Mathematics Contests 2016

Analysis and Differential Equations

Team

Please solve 5 out of the following 6 problems.

1. Let $D \subset \mathbf{R}^d, d \geq 2$ be a compact convex set with smooth boundary ∂D so that the origin belongs to the interior of D. For every $x \in \partial D$ let $\alpha(x) \in (0, \infty)$ be the angle between the position vector x of the outer normal vector $n(x)$. Let ω_d be the surface area of the unit sphere in $\mathbf{R}^d$. Compute:

$$\frac{1}{\omega_d} \int_{\partial D} \frac{\cos(\alpha(x))}{|x|^{d-1}} d\sigma(x)$$

where $d\sigma$ denotes the surface measure on ∂D.

2. Let $p > 0$ and suppose $f_n, f \in L^p[0,1]$ and $||f_n - f||_p = (\int_0^1 |f_n(x) - f(x)|^p dx)^{\frac{1}{p}} \to 0$ as $n \to \infty$.

a) Show that for every $\epsilon > 0$,

$$\lim_{n\to\infty} m(\{x \in [0,1] | |f_n(x) - f(x)| > \epsilon\}) = 0.$$

Here m is the Lebesgue measure.

b) Show that there exists a subsequence f_{n_j} such that $f_{n_j}(x) \to f(x)$ for almost every $x \in [0,1]$.

3. 1) Let f be a holomorphic function on the unit disk $D = \{z \in \mathbf{C} | |z| < 1\}$ except 0. Assume $f \in L^2(D), i.e. \int_D |f(z)|^2 dz d\bar{z} < \infty$, then 0 is a removable singularity.

2) Let f_n be a sequence of holomorphic functions over a domain $\Omega \subset C$ converging to f uniformly on any compact subset of Ω, does the sequence of its derivatives f_n' also have this property?

4. Consider the torus $T^2 = \mathbf{C}/\Lambda, \Lambda = \{m + in | m, n \in \mathbf{Z}\}$, i.e. $z_1, z_2 \in \mathbf{C}$ are equivalent if and only if there are integers m, n such that $z_2 = z_1 + m + in$ and T^2 are the space of equivalent classes. Show that the group of holomorphic automorphisms of T^2 is $SL(2, \mathbf{Z})$ of 2 x 2 integer matrices of determinant 1.

5. Let $\{e_n\}$ be an orth-normal basis of l_2 of square integrable functions over a circle. Let $A : l_2 \to l_2, Ae_1 = 0, Ae_n = \frac{e_{n-1}}{n-1}, n > 1$ be a linear operator. Show that A is an compact operator and A has no eigenvectors. What are the spectrum of A?

6. If $M = [0,1]$ is the unit interval, the heat kernel on M can be written

$$p(x, y, t) = \Sigma_k \phi_k(x) \phi_k(y) e^{\lambda_k t},$$

where $\{\lambda_k\}$ is an enumeration of the eigenvalues of the $\Delta = \frac{d^2}{dx^2}$ on M and $\{\phi_k\}$ are the corresponding eigenfunctions which vanish on ∂M.

i) Calculate $\{\lambda_k\}$ and the corresponding eigenfunctions.
ii) Prove that $|p(x, y, t)| \leq Ct^{-1/2}$, for all x, y, and $0 < t < 1$.
iii) What is the exponential rate of decay of $p(x, y, t)$ as $t \to \infty$, i.e. compute:

$$\lim_{t\to\infty} \log(p(x, y, t)).$$

S.-T. Yau College Student Mathematics Contests 2016

Probability and Statistics

Team (5 problems)

Problem 1. For a random walk process on the complete infinite binary tree (see Fig 1.) starting from root (i.e. level 0), we assume that the object moves to the neighbor nodes with equal probability. Let X_n denote the level number at time $= n$. Please prove that

$$\mathbb{E}X_n \leq 1/3n + 4/3$$

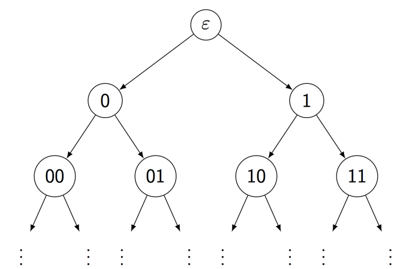


Fig 1.

Problem 2. The goal is to show the concentration inequality for the median of mean estimator. We divide the problem into three simple steps.

1. Let X be a random variable with $\mathbb{E}X = \mu < \infty$ and $\mathrm{Var}(X) = \sigma^2 < \infty$. Suppose we have m i.i.d. random samples $\{X_i\}_{i=1}^m$. Let $\hat{\mu}_m = \frac{1}{m}\sum_{i=1}^m X_i$ from X. Show that
$$P\big(|\widehat{\mu}_m - \mu| \geq 2\sigma\sqrt{\frac{1}{m}}\big) \leq \frac{1}{4}.$$

2. Given k i.i.d. Bernoulli random variables $\{B_j\}_{j=1}^k$ with $\mathbb{E}B_j = p < \frac{1}{2}$. Use the moment generating function of B_j, i.e., $\mathbb{E}(\exp(tB_j))$, to show that
$$P\big(\frac{1}{k}\sum_{j=1}^k B_j \geq \frac{1}{2}\big) \leq (4p(1-p))^{\frac{k}{2}}.$$

3. Suppose we have n i.i.d. random samples $\{X_i\}_{i=1}^n$ from a population with mean μ and variance σ^2. For any positive integer k, we randomly and uniformly divide all the samples into k subsamples, each having size $m = n/k$ (for simplicity, we assume n is always divisible by k). Let $\widehat{\mu}_j$ be the sample average of the j^{th}

subsample and $\widetilde{m}$ be the median of $\{\widehat{\mu}_j\}_{j=1}^k$. Apply the previous two results to show that

$$P\Big(|\widetilde{m}-\mu| \geq 2\sigma\sqrt{\frac{k}{n}}\Big) \leq \Big(\frac{\sqrt{3}}{2}\Big)^k.$$

Hint: Consider the Bernoulli random variable $B_j = \mathbb{1}\{|\widehat{\mu}_j - \mu| \geq 2\sigma\sqrt{\frac{k}{n}}\}$ for $j = 1, ..., k$.

Problem 3. (a) Let $N \geq 2$ be an integer, and let X be a random variable taking values in $\{0, 1, 2, \ldots\}$ such that $P\{X \equiv k \,(\mathrm{mod}\, N)\} = \frac{1}{N}$ for all $k \in \{0, 1, \ldots, N-1\}$. Compute $\mathbb{E}(e^{i(2\pi m)X/N})$ (with $i = \sqrt{-1}$) for all integers $m \geq 1$.

(b) A game for N players (numbered as 0, 1, 2, …, $N-1$) is as follows: Each player independently shows a random number of fingers (uniformly chosen from $\{0, 1, 2, 3, 4, 5\}$); if S denotes the total number of fingers shown, then the player number $S \bmod N$ is declared to be the winner of the game.

Find all N such that the players have equal chance to win the game.

Problem 4. Let X_1, X_2, … be independent and identically distributed real-valued random variables. Prove or disprove: If $\limsup_{n\to\infty} \frac{|X_n|}{n} \leq 1$ almost surely, then $\sum_{n=1}^{\infty} P(|X_n| \geq n) < \infty$.

Problem 5. Choose, at random, 2016 points on the circle $x^2 + y^2 = 1$. Interpret them as cuts that divide the circle into 2016 arcs. Compute the expected length of the arc that contains the point $(1, 0)$. How about the variance.

S.-T. Yau College Student Mathematics Contests 2016

Geometry and Topology

Team

Please solve 5 out of the following 6 problems.

1. Show that $\mathbb{CP}^{2n}$ does not cover any manifold except itself.

2. Let X be a topological space and $p \in X$. The *reduced suspension* ΣX of X is the space obtained from $X \times [0,1]$ by contracting $(X \times \{0,1\}) \cup (\{p\} \times [0,1])$ to a point. Describe the relation between the homology groups of X and ΣX.

3. State and prove the Frobenius Theorem on a differentiable manifold.

4. Show that all geodesics on the sphere S^n are precisely the great circles.

5. Let M be an n-dimensional Riemannian manifold. Denote by R and K_M the curvature tensor and sectional curvature of M. If $a \leq K_M \leq b$ at a point $x \in M$, then, at this point,

$$R(e_1, e_2, e_3, e_4) \leq \frac{2}{3}(b-a)$$

for all orthonormal four-frames $\{e_1, e_2, e_3, e_4\} \subset T_xM$.

6. Let M be a closed minimal hypersurface with constant scalar curvature in S^{n+1}. Denote by S the squared length of the second fundamental form of M. Show that $S = 0$, or $S \geq n$.

S.-T. Yau College Student Mathematics Contests 2016

Algebra and Number Theory

Team

This test has 5 problems and is worth 100 points. Carefully justify your answers.

Problem 1 (20 points)**.** Find all real orthogonal 2×2 matrices k with the following property: There is an upper triangular 2×2 real matrix b with all diagonal entries being positive numbers such that kb is a positive definite symmetric matrix.

Problem 2 (20 points)**.** For $x \in \mathbb{Z}$ and $k \geq 0$, define the binomial coefficients

$$\binom{x}{k} = \frac{x(x-1)\cdots(x-k+1)}{k!}, \qquad \binom{x}{0} = 1.$$

(a) (6 points) Show that $x \in \mathbb{Z} \implies \binom{x}{k} \in \mathbb{Z}$.
(b) (6 points) Show that every function $f\colon \mathbb{Z}_{\geq 0} \to \mathbb{Z}$ can be expressed as $f(x) = \sum_{k=0}^{\infty} a_k \binom{x}{k}$, where $a_k \in \mathbb{Z}$ are uniquely determined by f.
(c) (8 points) Define

$$\phi_k(x) = \binom{x + \lfloor k/2 \rfloor}{k}.$$

Show that every function $f\colon \mathbb{Z} \to \mathbb{Z}$ can be expressed as $f(x) = \sum_{k=0}^{\infty} a_k \phi_k(x)$, where $a_k \in \mathbb{Z}$ are uniquely determined by f.

Problem 3 (20 points)**.** Let K be the splitting field of the polynomial

$$x^4 - x^2 - 1.$$

(a) (10 points) Show that the Galois group of K over $\mathbb{Q}$ is isomorphic to the dihedral group D_4. Here we adopt the convention that D_4 is the group of symmetries of a square and has order 8.
(b) (10 points) Determine the lattice of subfields of K: Find all subfields of K and describe the partial order induced by inclusion.

Problem 4 (20 points)**.** Let G be a (not necessarily finite) group and let F be a field of characteristic $\neq 2$. Let $V \neq 0$ be an **indecomposable** finite-dimensional linear representation of G over F. Let $R = \mathrm{End}_F(V)^G$ be the ring of G-equivariant endomorphisms of V.
(a) (5 points) Prove the following form of Fitting's lemma: Every element of R is either invertible or nilpotent.
(b) (5 points) Deduce that the set $I \subseteq R$ of non-invertible elements is a two-sided ideal and the quotient R/I is a division algebra over F.
(c) (5 points) We say that V is *orthogonal* if there **exists** a G-invariant nondegenerate symmetric bilinear form on V. We say that V is *symplectic* if there **exists** a G-invariant nondegenerate alternating bilinear form on V. Deduce that if there exists a G-invariant nondegenerate bilinear form on V, then V is orthogonal or symplectic.

(d) (5 points) Assume that F is algebraically closed. Deduce from (b) that V cannot be both orthogonal and symplectic.

Problem 5 (20 points)**.**

(a) (5 points) Let G be a finite group. Let $x_1, \dots, x_h$ be representatives of the conjugacy classes of G. Let $n_i = \#\mathrm{Cent}_G(x_i)$ be the cardinality of the centralizer of x_i. Prove the identity

$$1 = \sum_{i=1}^{h} \frac{1}{n_i}.$$

(b) (10 points) Deduce that for any integer $h \geq 1$, there exist only finitely many isomorphism classes of finite groups with exactly h conjugacy classes.

(c) (5 points) Find all the finite groups with exactly 3 conjugacy classes.

S.-T. Yau College Student Mathematics Contests 2016

Applied Math. and Computational Math.

Team (5 problems)

Problem 1. For solving the following partial differential equation

(1) $$u_t + u_x = 0, \qquad -\infty \le x \le \infty$$

with compactly supported initial condition, we consider the following one-step, three-point scheme on a uniform mesh $x_j = j\Delta x$ with spatial mesh size Δx:

(2) $$u_j^{n+1} = au_j^n + bu_{j-1}^n + cu_{j-2}^n, \qquad j = \cdots, -1, 0, 1, \cdots$$

where a, b, c are constants which may depend on the mesh ratio $\lambda = \frac{\Delta t}{\Delta x}$. Here Δt is the time step, and u_j^n approximates the exact solution at $u(x_j, t^n)$ with $t^n = n\Delta t$.

(1) Find the constants a, b, c such that the scheme (2) is second order accurate.

(2) Find the CFL number λ_0 such that the scheme (2), with the constants determined by the step above, is stable in L^2 under the time step restriction $\lambda \le \lambda_0$.

(3) If the PDE (1) is defined on $(0, \infty)$ with an initial condition compactly supported in $(0, \infty)$ and a boundary condition $u(0, t) = g(t)$, how would you modify the scheme (2) so that it can be applied? Can you prove the stability and accuracy of your modified scheme?

Problem 2. Inverse problem. Answer the famous Mark Kac's equation: "can you hear the shape of drum?" for the special case.

Consider the one-dimensional oscillator $\ddot{x} = -u'(x)$ with symmetric potential $u(-x) = u(x)$, $u(0) = u'(0) = 0$, $u'(x) > 0$ for $x > 0$, $\lim_{x\to\infty} u(x) = \infty$. Denote the inverse function of $y = u(x)$, $x \ge 0$ as $x = u^{-1}(y) = \phi(y)$.

(a) For any solution $x(t)$, show there is a conservation of energy

$$\frac{\dot{x}^2(t)}{2} + u(x(t)) \equiv e$$

where e is a constant.

(b) For any energy $e > 0$, find a periodic solution with total energy e. Show that the period is given by

$$P(e) = 2\sqrt{2}\int_0^{x_{max}} \frac{dx}{\sqrt{e - u(x)}}, \qquad x_{max} = \phi(e) > 0\,.$$

(c) Show that

$$\phi(z) = \frac{1}{2\pi\sqrt{2}}\int_0^z \frac{P(e)\,de}{\sqrt{z-e}}\,.$$

(d) In the case of iso-chronous $P(e) \equiv 2\pi$, show that $\phi(z) = \sqrt{2z}$. Then you have $u(x) = \frac{1}{2}x^2$, $x(t) = a\cos(t) + b\sin(t)$, the famous harmonic oscillator.

Problem 3. The following statement *informally* means that if a system of homogeneous equations with integer coefficients has a nontrivial solution then it has an integer solutions with reasonably small components. It is required in many applications.
Let $A = (a_{ij})_{i,j=1}^{m,n}$ be an $m \times n$ matrix of rank $r \le n-1$ with integer entries of size at most H, that is,

$$|a_{ij}| \le H, \quad 1 \le i \le m,\ 1 \le j \le n.$$

Prove that there is an integer **non-zero** vector $\mathbf{x} = (x_1, \dots, x_n) \in \mathbb{Z}^n$ such that $A\mathbf{x} = \mathbf{0}$ and

$$\|\mathbf{x}\|_\infty \le (2nH)^{n-1}$$

where $\|\mathbf{x}\|_\infty = \max_{1 \le i \le n} |x_i|$.

Problem 4. This problem considers an iterative scheme

$$x_{k+1} = x_k + \beta_k p_k$$

for the linear system $Ax = b$, where $A \in \mathbb{R}^{n \times n}$ is a given $n \times n$ non-singular matrix and $b \in \mathbb{R}^n$ is a given vector. In the above scheme, x_k denotes the approximate solution at the k-th iteration, β_k is a scalar and $p_k \in \mathbb{R}^n$ is a search direction. If x_k is given, the above scheme will determine x_{k+1} so that the residual $r_{k+1} := b - Ax_{k+1}$ is the smallest possible with respect to the 2-norm.

(1) Determine β_k.
(2) Prove that the residual r_{k+1} is orthogonal to Ap_k with respect to the usual inner-product.
(3) Prove that the residuals satisfy

$$\|r_{k+1}\| \le \|r_k\| \sin(\alpha)$$

where α is the angle between r_k and Ap_k, and $\|\cdot\|$ denotes the 2-norm.
(4) Assume that the inner product of r_k and Ap_k is non-zero. Will the above scheme always converge?
(5) Assume that A is positive definite. We take the search direction $p_k = r_k$. Show that the above scheme converges for any initial guess x_0.

Problem 5. Let $f : \mathbb{R}^n \to \mathbb{R}$ be convex and in C^1. Suppose f has a local minimum x^*.

(1) Must this local minimum x^* be a global minimum?
(2) Consider the following backward gradient method: starting from any $x^0 \in \mathbb{R}^n$, define

$$x^k = x^{k-1} - t\nabla f(x^k), \quad k \ge 1,$$

where $t > 0$ is a fixed step size. Do you need any condition on t to guarantee $\{f(x^k)\}$ converge? Prove your convergence argument, if $\{f(x^k)\}$ converges.
(3) Suppose f is strongly convex, that is, $\exists m > 0$ such that $\langle \nabla f(x) - \nabla f(y), x-y \rangle \ge m\|x-y\|^2$. Under this additional condition, show that $\{x^k\}$ converges.

S.-T. Yau College Student Mathematics Contests 2017

Analysis and Differential Equations

Individual

Please solve 5 out of the following 6 problems.

1. Suppose f is integrable on $[-\pi, \pi]$, prove that $\Sigma_{n=-\infty}^{\infty} a_n r^{|n|} e^{inx}$ tends to $f(x)$ for a.e. x, as $r \to 1, r < 1$. Here $a_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) e^{-inx} dx$.

2. Let H be a Hilbert space equipped with an inner product $(.,.)$ and a norm $||.|| = (.,.)^{\frac{1}{2}}$. A sequence $\{f_k\}$ is converge to $f \in H$ if $||f_k - f|| \to 0$. A sequence $\{f_k\} \subset H$ is said converge weakly to $f \in H$ if $(f_k, g) \to (f, g)$ for any $g \in H$. Prove the following statements:

a) $\{f_k\}$ converges to f if and only if $||f_k|| \to ||f||$ and $\{f_k\}$ converges weakly to f.

b) If H is a finite dimensional Hilbert space, then the weak convergence implies convergence. Give a counter example to show that weak convergence does not necessarily imply convergence in an infinite dimensional Hilbert space.

3. Let $f : \mathbf{C}/\{0\} \to \mathbf{C}$ be a holomorphic function and

$$|f(z)| \leq |z|^2 + \frac{1}{|z|^{1/2}},$$

for z near 0. Determine all such functions.

4. Find a conformal mapping which maps the region $\{z||z-i| < \sqrt{2}, |z+i| < \sqrt{2}\}$ onto the unit disk.

5. If E is a compact set in a region Ω, prove that there exists a constant $M > 0$, depending only on E and Ω, such that every positive harmonic function $u(z)$ in Ω satisfies $u(z_2) \leq M u(z_1)$ for any two points $z_1, z_2 \in E$.

6. 1) For any bounded domain $\Omega \subset \mathbf{R}^n$, there exists a smallest constant $C(\Omega)$, such that

$$\int_\Omega |u|^2 dx \leq C(\Omega) \int_\Omega \Sigma_{i=1}^n |\frac{\partial u}{\partial x_i}|^2 dx$$

for every function $u \in H_0^1(\Omega) = \overline{C_0^\infty(\Omega)} \subset H^1(\Omega)$, where $C_0^\infty(\Omega)$ is the space of smooth functions over Ω and vanishing on boundary of Ω and $H^1(\Omega)$ is the Banach space of functions $u \in L^2(\Omega), \nabla u \in L^2(\Omega)^{\otimes n}$ with the norm:

$$||u||^2_{H^1(\Omega)} = ||u||^2_{L^2(\Omega)} + ||\nabla u||^2_{L^2(\Omega)^{\otimes n}}$$

$$= \int_\Omega (|u|^2 + \Sigma_{i=1}^n |\frac{\partial u}{\partial x_i}|^2) dx.$$

$H_0^1(\Omega)$ is the completion of $C^\infty(\Omega)$ in $H^1(\Omega)$ with the above norm.

2) Let $\Pi = \{(x, y)|0 < x < a, 0 < y < b\}$, show that $C(\Pi) \geq \frac{a^2 b^2}{\pi^2(a^2+b^2)}$.

S.-T. Yau College Student Mathematics Contests 2017

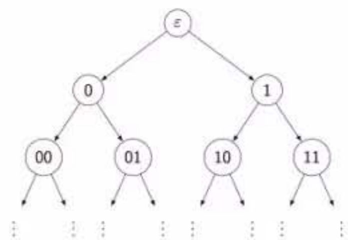
Probability and Statistics

Individual (5 problems)

Problem 1. A box contains 750 red balls and 250 blue balls. Repeatedly pick a ball uniformly at random from the box and remove it until all remaining balls have a single color. (Note: no replacement).

Please find integer m such that the expectation value for the total number of the remaining balls $\in [m, m+1]$

Problem 2. Suppose a number $X_0 \in \{1, -1\}$ at the root of a binary tree



is propagated away from the root as follows. The root is the node at level 0. After obtaining the 2^h numbers at the nodes at level h, each number at level $h+1$ is obtained from the number adjacent to it (at level h) by flipping its sign with probability $p \in (0, 1/2)$ independently.

Let X_h be the average of the 2^h values received at the nodes at level h. Define the *signal-to-noise ratio* at level h to be

$$R_h := \frac{\left(\mathbb{E}[X_h \mid X_0 = 1] - \mathbb{E}[X_h \mid X_0 = -1]\right)^2}{Var[X_h|X_0 = 1]}.$$

Find the threshold number p_c such that R_h converges to 0 if $p \in (p_c\,,\, 1/2)$ and diverges if $p \in (0,\, p_c)$, as $h \to \infty$.

Problem 3. Consider the space representing an infinite sequence of coin flips, namely $\Omega := \{H, T\}^\infty$, (H: head, T: tail) with the associated σ-field $\mathcal{F}$ generated by finite dimensional rectangles. For $0 \leq p \leq 1$, denote by $\mathbb{P}_p$ the probability measure on $(\Omega, \mathcal{F})$ corresponding to flipping a coin an infinite number of times with probability of H being p and probability of T being $q = 1 - p$ at each flip.

Show that for each $p \in [0, 1]$, there exists A_p such that

$$\mathbb{P}_p(A_p) > 1/2$$

and for any $p' \neq p$, $p' \in [0, 1]$

$$\mathbb{P}_{p'}(A_p) < 1/2$$

Problem 4. Let $G := G(n, p)$ be a random graph with n vertices where each possible edge has probability p of existing. The existence of the edges are independent to each other. With G, we say $A \subset \{1, 2, \cdots, n\}$ is a fully connected set if and only if

$$i, j \in A \implies i - th \text{ and } j - th \text{ vertices are (directly) connected with an edge in } G$$

Define T as the size of the largest fully connected set

$$T := \max\{|A| : \text{ A is a fully connected set}\}$$

Let's fix $p \in (0, 1)$, please prove that

$$\lim_{n \to \infty} \mathbb{P}\left(\frac{T}{2\log_{\frac{1}{p}} n} \leq 1 + \epsilon\right) = 1, \quad \forall \epsilon > 0,$$

and

$$\lim_{n \to \infty} \mathbb{P}\left(\frac{T}{\sqrt{2\log_{\frac{1}{p}} n}} \geq 1 - \epsilon\right) = 1, \quad \forall \epsilon > 0,$$

Hint:

$$\mathbb{P}(T = n) = p^{\binom{n}{2}} = p^{n(n-1)/2}$$

Problem 5. Consider a population of constant size $N + 1$ that is suffering from an infectious disease. We can model that spread of the disease as Markov process. Let $X(t)$ be the number of healthy individuals at time t and suppose that $X(0) = N$. We assume that if $X(t)$

$$\lim_{h \to 0} \frac{1}{h}\mathbb{P}(X(t + h) = n - 1 | X(t) = n)) = \lambda n (N + 1 - n)$$

For $0 \leq s \leq 1$, $0 \leq t$, define

$$G(s, t) := \mathbb{E}(s^{X(t)})$$

Please find a non-trivial partial differential equation for $G(s, t)$, which involves $\partial_t G$.

S.-T. Yau College Student Mathematics Contests 2017

Geometry and Topology

Individual

Please solve 5 out of the following 6 problems.

1. Let M be a smooth, compact, oriented n-dimensional manifold. Suppose that the Euler characteristic of M is zero. Show that M admits a nowhere vanishing vector field.

2. Let $S^2 \xleftarrow{q_1} S^2 \vee S^2 \xrightarrow{q_2} S^2$ be the maps that crush out one of the two summands. Let $f : S^2 \to S^2 \vee S^2$ be a map such that $q_i \circ f : S^2 \to S^2$ is a map of degree d_i. Compute the integral homology groups of $(S^2 \vee S^2) \cup_f D^3$. Here D^3 is the unit solid ball with boundary S^2.

3. Let X and Y be smooth vector fields on a smooth manifold. Prove that the Lie derivative satisfies the identity

$$L_X Y = [X, Y].$$

4. State and prove the Liouville formula for the geodesic curvature κ_g along a regular curve on a smooth surface in $\mathbb{R}^3$.

5. On a Riemannian manifold, let F be the set of smooth functions f on M with $|\mathrm{grad} f| \leq 1$. For any x, y in the manifold, show that

$$d(x, y) = \sup\{|f(x) - f(y)| : f \in F\}.$$

6. Let M be an n-dimensional oriented closed minimal submanifold in an $(n+p)$-dimensional unit sphere S^{n+p}. Denote by K_M the sectional curvature of M. Prove that if $K_M > \dfrac{p-1}{2p-1}$, then M is the great sphere S^n.

S.-T. Yau College Student Mathematics Contests 2017

Algebra and Number Theory

Individual

This test has 5 problems and is worth 100 points. Carefully justify your answers.

Problem 1 (20 points). Let $\mathbb{Q}_p$ denote the field of p-adic numbers and let $\mathbb{Z}_p$ denote the ring of p-adic integers (p is a prime number).

(a) (5 points) Show that for every integer $k \geq 0$, $(p^{-k}\mathbb{Z}_p)/\mathbb{Z}_p \cong \mathbb{Z}/p^k\mathbb{Z}$ as abelian groups.

(b) (5 points) Determine the endomorphism ring of the abelian group $(p^{-k}\mathbb{Z}_p)/\mathbb{Z}_p$ ($k \geq 0$).

(c) (5 points) Determine the endomorphism ring of the abelian group $\mathbb{Q}_p/\mathbb{Z}_p$.

(d) (5 points) Determine the endomorphism ring of the abelian group $\mathbb{Q}/\mathbb{Z}$.

Problem 2 (20 points). Let A be a finite abelian group and let $\phi : A \to A$ be an endomorphism. Put

$$A_{\text{nil}} := \{x \in A \mid \phi^k(x) = 0 \text{ for some } k \geq 1\}.$$

(a) (15 points) Show that there is a subgroup A_0 of A such that ϕ restricts to an automorphism of A_0 and $A = A_0 \oplus A_{\text{nil}}$.

(b) (5 points) Show that such a subgroup is unique.

Problem 3 (20 points). Let L/F be a Galois field extension, not necessarily finite. Let $x \in L$.

(a) (6 points) Show that the set $\mathcal{P}$ of subextensions of L/F not containing x has a maximal element E. Let K/E be a nontrivial finite extension contained in L. Show that $x \in K$.

(b) (6 points) Let K' be the Galois closure of K/E in L. Show that there exists $g \in G = \mathrm{Gal}(K'/E)$ such that $gx \neq x$.

(c) (8 points) Deduce that K/E is a cyclic Galois extension.

Problem 4 (20 points). The goal of this problem is to prove the Chevalley–Warning theorem. Let p be a prime number and q a power of p.

(a) (8 points) Let $0 \leq a < q-1$ be an integer. Show that $S(X^a) := \sum_{x \in \mathbb{F}_q} x^a$ equals 0. Here we adopt the convention $x^0 = 1$ in $\mathbb{F}_q$ even for $x = 0$.

(b) (12 points) Let $f_1, \ldots, f_m \in \mathbb{F}_q[X_1, \ldots, X_n]$ be polynomials in n variables satisfying

$$\sum_{i=1}^{m} \deg(f_i) < n.$$

Show that $P = \prod_{i=1}^{m}(1 - f_i^{q-1})$ satisfies

$$S(P) := \sum_{(x_1,\ldots,x_n) \in \mathbb{F}_q^n} P(x_1, \ldots, x_n) = 0.$$

Deduce that p divides the cardinality of the set

$$V = \{(x_1, \ldots, x_n) \in \mathbb{F}_q^n \mid f_i(x_1, \ldots, x_n) = 0 \quad \forall i\}.$$

Problem 5 (20 points). In this problem, all matrices are $n \times n$ with complex entries. Let U and V be matrices such that $UV \neq VU$. Assume that U is diagonalizable and commutes with VUV^{-1}.

(a) (10 points) For $\lambda, \mu \in \mathbb{C}$, let

$$E_{\lambda,\mu} = \{x \in \mathbb{C}^n \mid Ux = \lambda x, \quad VUV^{-1}x = \mu x\}.$$

Show that there exist couples $(\lambda_1, \mu_1) \neq (\lambda_2, \mu_2)$, satisfying $\lambda_i \neq \mu_i$ and $E_{\lambda_i,\mu_i} \neq 0$ for $i = 1, 2$.

(b) (10 points) For a matrix A, we define $\mathrm{N}(A) := \mathrm{tr}(A^*A)$, where $A^* = \bar{A}^T$ is the conjugate transpose of A. Assume that U and V are unitary (namely, $U^*U = V^*V$ is the identity matrix). Deduce that $\mathrm{N}(1 + V) \geq 4$.

S.-T. Yau College Student Mathematics Contests 2017

Applied Math. and Computational Math.

Individual (5 problems)

1. The Chebyshev polynomial of the first kind is defined on $[-1,1]$ by

$$T_n(x) = \cos(n \arccos x).$$

Prove: The envelope for the extremals of $T_{n+1}(x) - T_{n-1}(x)$ forms an ellipse.

2. Consider a fixed point iteration

$$x_n = g(x_{n-1}),$$

where $g : \mathbb{R} \to \mathbb{R}$ is a smooth function. Suppose this fixed point method does converge to a fixed point x^*. The Steffensen algorithm is an acceleration method to find x^* which reads

$$\hat{x}_n = x_{n-2} - \frac{(x_{n-1} - x_{n-2})^2}{x_n - 2x_{n-1} + x_{n-2}}.$$

or

$$x_{n+1} = G(x_n)$$

where

$$G(x) = x - \frac{(g(x) - x)^2}{g(g(x)) - 2g(x) + x}.$$

(a) Show that the Steffensen algorithm $\{x_k\}$ converges quadratically.

(b) Can you extend this method to two dimensions?

3. We consider a piecewise smooth function

$$f(x) = \begin{cases} f_1(x), & x \le 0, \\ f_2(x), & x > 0 \end{cases}$$

where $f_1(x)$ is a C^∞ function on $(-\infty, 0]$ and $f_2(x)$ is a C^∞ function on $[0,\infty)$, but $f_1(0) \neq f_2(0)$. Suppose $p(x)$ is a k-th degree polynomial ($k \ge 1$) interpolating $f(x)$ at $k+1$ equally-spaced grid points x_j, $j = 0, 1, 2, \cdots, k$ with $x_i < 0 < x_{i+1}$ for some i between 0 and $k-1$. Prove that, when the grid size $h = x_{j+1} - x_j$ is small enough, $p'(x) \neq 0$ for $x_i \le 0 \le x_{i+1}$, that is, $p(x)$ is monotone in the interval $[x_i, x_{i+1}]$. (**Hint: first prove the case when $f_1(x) = c_1$, $f_2(x) = c_2$ and $c_1 \neq c_2$ are two constants.**)

4. Let $b \in \mathbb{R}^n$. Suppose $A \in M_{n\times n}(\mathbb{R})$ and $B \in M_{n\times n}(\mathbb{R})$ are two $n \times n$ matrices. Let A to be non-singular.

(a) Consider the iterative scheme: $Ax^{k+1} = b - Bx^k$.
State and prove the necessary and sufficient condition for the iterative scheme to converge.

(b) Suppose the spectral radius of $A^{-1}B$ satisfies $\rho(A^{-1}B) = 0$. Prove that the iterative scheme converges in n iterations.

(c) Consider the following iterative scheme:

$$x^{(k+1)} = \omega_1 x^{(k)} + \omega_2(c_1 - Mx^{(k)}) + \omega_3(c_2 - Mx^{(k)}) + ... + \omega_k(c_{k-1} - Mx^{(k)})$$

where M is symmetric and positive definite, $\omega_1 > 1$, $\omega_2, ..., \omega_k > 0$ and $c_1, ..., c_{k-1} \in \mathbb{R}^n$. Deduce from (a) that the iterative scheme converges if and only if all eigenvalues of M (denote it as $\lambda(M)$) satisfies:

$$(\omega_1 - 1)/(\sum_{i=2}^{k} \omega_i) < \lambda(M) < (\omega_1 + 1)/(\sum_{i=2}^{k} \omega_i).$$

(d) Let A be non-singular. Now, consider the following system of iterative scheme (*):

$$Ax_1^{(k+1)} = b_1 - Bx_2^{(k)}, \quad Ax_2^{(k+1)} = b_2 - Bx_1^{(k)}$$

Find and prove the necessary and sufficient condition for the iterative scheme (*) to converge.

For the iterative scheme (**):

$$Ax_1^{(k+1)} = b_1 - Bx_2^{(k)}, \quad Ax_2^{(k+1)} = b_2 - Bx_1^{(k+1)}$$

Find and prove the necessary and sufficient condition for the iterative scheme (**) to converge. Compare the rate of convergence of the iterative schemes (*) and (**).

5. Consider the differential equation

$$-u'' + \alpha u = f, \ x \in (0, 1).$$

Here, prime denotes for d/dx and α is a constant. We consider a mixed boundary condition

$$u(0) = 0, \ u'(1) - bu(0) = 0.$$

This equation is approximated by a standard finite difference method:

$$\frac{-U_{j-1} + 2U_j - U_{j+1}}{h^2} + \alpha U_j = f_j, j = 1, ..., N-1.$$

Here, N is the number of grid points, $h = 1/N$ is the mesh size, U_j is the approximate solution at $x_j := jh$, and $f_j = f(x_j)$. The noundary condition is approximated by

$$U_0 = 0, \ \frac{U_N - U_{N-1}}{h} - bU_N = 0.$$

The resulting linear system is $AU = F$ with

$$\begin{bmatrix} \beta & -1 & 0 & \cdots & & & \\ -1 & \beta & -1 & \cdots & & & \\ & & & \ddots & & & \\ & & & & -1 & \beta & -1 \\ & & & & 0 & -1 & 1-bh \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \\ \vdots \\ U_{N-1} \\ U_N \end{bmatrix} = \begin{bmatrix} h^2 f_1 \\ h^2 f_2 \\ \vdots \\ h^2 f_{N-1} \\ 0 \end{bmatrix}$$

where $\beta = 2 + \alpha h^2$.

$$u_t + a u_x = 0, \ a > 0.$$

We discretize this PDE by For solving the following partial differential equation

(1) $$u_t + f(u)_x = 0, \qquad 0 \le x \le 1$$

where $f'(u) \ge 0$, with periodic boundary condition, we can use the following semi-discrete upwind scheme

(2) $$\frac{d}{dt} u_j + \frac{f(u_j) - f(u_{j-1})}{\Delta x} = 0, \qquad j = 1, 2, \cdots, N,$$

with periodic boundary condition

(3) $$u_0 = u_N,$$

where $u_j = u_j(t)$ approximates $u(x_j, t)$ at the grid point $x = x_j = j\Delta x$, with $\Delta x = \frac{1}{N}$.

(a) Prove the following L^2 stability of the scheme

(4) $$\frac{d}{dt} E(t) \le 0$$

where $E(t) = \sum_{j=1}^{N} |u_j|^2 \Delta x$.

(b) Do you believe (4) is true for $E(t) = \sum_{j=1}^{N} |u_j|^{2p} \Delta x$ for *arbitrary* integer $p \ge 1$? If yes, prove the result. If not, give a counterexample.